MATH 403 LECTURE NOTE
WEEK 4

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1. Theorem of Menelaus (Sec 1.12)

Recall that $P \in \ell_{AB}$ if and only if there exist a, b such that $P = aA + bB$ and $a + b = 1$.

Definition 1.1. *Three points A, B, C are called* collinear *if there exists a line ℓ such that $A, B, C \in \ell$.*

Theorem 1.2. *Three distinct points A, B, C are collinear if and only if there exist a, b, c not all zero such that $a + b + c = 0$ and $aA + bB + cC = O$.*

Proof. If A, B, C are collinear, then $C \in \ell_{AB}$. Thus, there exists $r \in \mathbb{R}$ such that $C = (1 - r)A + rB$. Let $a = 1 - r, b = r, c = -1$, then $a + b + c = 0$ and $aA + bB + cC = O$.

Suppose $aA + bB + cC = O$ with $a + b + c = 0$. Since a, b, c are not all zero, we assume $c \neq 0$ without loss of generality. Then, $C = -\frac{a}{c}A - \frac{b}{c}B$. Since $-\frac{a}{c} - \frac{b}{c} = 1$, we have $C \in \ell_{AB}$. ∎

Theorem 1.3. *Let A, B, C form a triangle. Let $A' \in \ell_{BC}, B' \in \ell_{CA}, C' \in \ell_{AB}$ be distinct from A, B, C. Then, A', B', C' are collinear if and only if*

$$\frac{A' - B}{A' - C} \cdot \frac{B' - C}{B' - A} \cdot \frac{C' - A}{C' - B} = 1$$

Proof. Assume that A', B', C' are collinear. Then, there are $r, s, t \in \mathbb{R}$ such that

$$A' = (1 - r)B + rC, \quad B' = (1 - s)C + sA, \quad C' = (1 - t)A + tB.$$

Since A', B', C' are distinct from A, B, C, we have $r, s, t \neq 0, 1$. Note that $(B' - C) = s(A - C)$ and $(A' - C) = (1 - r)(B - C)$. Then,

$$\begin{aligned} C' - C &= (1 - t)(A - C) + t(B - C) \\ &= \frac{1 - t}{s}(s(A - C)) + \frac{t}{1 - r}((1 - r)(B - C)) \\ &= \frac{1 - t}{s}(B' - C) + \frac{t}{1 - r}(A' - C). \end{aligned}$$

Since A', B', C' are collinear, the sum of coefficients is 1, that is,

$$\frac{1 - t}{s} + \frac{t}{1 - r} = 1.$$

Solving for t, we get

$$t = \frac{(1 - r)(s - 1)}{r + s - 1}, \quad 1 - t = \frac{rs}{r + s - 1}.$$

(Why $r + s - 1 \neq 0$?) Then,

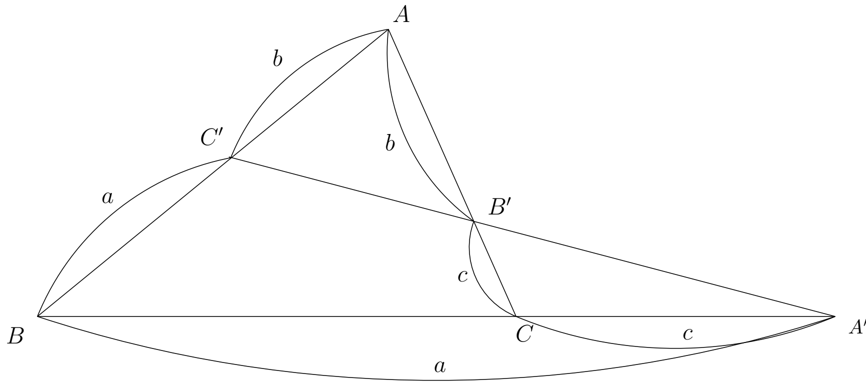
$$\frac{A' - B}{A' - C} \cdot \frac{B' - C}{B' - A} \cdot \frac{C' - A}{C' - B} = \frac{r}{r - 1} \cdot \frac{s}{s - 1} \cdot \frac{t}{t - 1} = \frac{r}{r - 1} \cdot \frac{s}{s - 1} \cdot \frac{(r - 1)(s - 1)}{rs} = 1.$$

Assume that the product of ratios equals to 1. In particular, we assume that

$$\frac{B' - C}{B' - A} = -\frac{c}{b}, \quad \frac{C' - A}{C' - B} = -\frac{b}{a},$$

then the assumption yields

$$\frac{A' - B}{A' - C} = \frac{a}{c}.$$



Solving for B', we get

$$\begin{aligned} B' &= \frac{c}{b+c}A + \frac{b}{b+c}C, \\ B' - B &= \frac{c}{b+c}(A - B) + \frac{b}{b+c}(C - B) \\ &= \frac{c}{b+c} \cdot \frac{a+b}{a}(C' - B) + \frac{b}{b+c} \cdot \frac{a-c}{a}(A' - B). \end{aligned}$$

Since

$$\frac{c}{b+c} \cdot \frac{a+b}{a} + \frac{b}{b+c} \cdot \frac{a-c}{a} = 1,$$

we conclude that A', B', C' are collinear. ∎

2. Theorem of Pappus (Sec 1.13)

Remark 1. *Recall that If $X, Y \in \mathbb{R}^2$ and $\ell_{OX} \neq \ell_{OY}$, then every $P \in \mathbb{R}^2$ can be uniquely written as $P = aX + bY$ for $a, b \in \mathbb{R}$. It follows from the uniqueness that*

$$aX + bY = O \Rightarrow a = b = 0$$

or

$$aX + bY = a'X + b'Y \Rightarrow a = a', b = b'.$$

The condition $\ell_{OX} \neq \ell_{OY}$ is equivaluent to $O \notin \ell_{XY}$.

Theorem 2.1 (Pappus)**.** *Let $\ell_{BB'}$, $\ell_{CC'}$ be two distinct lines through O with $X \in \ell_{BB'}$ and $Y \in \ell_{CC'}$. If $\ell_{BC'} \parallel \ell_{B'Y}$ and $\ell_{B'C} \parallel \ell_{XC'}$, then $\ell_{BC} \parallel \ell_{XY}$.*

Proof. Since $B', X \in \ell_{OB}$ and $C', Y \in \ell_{OC}$, there exist $m_1, m_2, n_1, n_2 \in \mathbb{R}$ such that

$$B' = m_1 B, \quad X = m_2 B, \quad C' = n_1 C, \quad Y = n_2 C.$$

Since $\ell_{BC'} \parallel \ell_{B'Y}$, there exists $a \in \mathbb{R}$ such that $Y - B' = a(C' - B)$. Similarly, we have $X - C' = b(B' - C)$ for some $b \in \mathbb{R}$. Thus,

$$\begin{aligned} Y = n_2 C = B' + a(C' - B) &= (m_1 - a)B + a n_1 C, \\ 0 &= (m_1 - a)B + (a n_1 - n_2)C. \end{aligned}$$

By the remark, we have $m_1 = a$ and $a n_1 = n_2$. Similarly, we have $n_1 = b$ and $m_2 = b m_1$. Therefore, $m_2 = n_2 = ab$ and

$$X - Y = ab(B - C).$$

∎

3. TRANSLATIONS (SEC 2.1)

Consider an assignment α from the set of points of the plane to itself. We call α a *map* or a *correspondence*. If α assigns a point X to Y, we use the notation $\alpha(X) = Y$. We say two maps α and β are *equal* if $\alpha(X) = \beta(X)$ for all $X \in \mathbb{R}^2$.

Definition 3.1. *(1) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called* one-to-one *if $\alpha(X) = \alpha(X')$ implies $X = X'$.*
(2) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called onto *if for every $Y \in \mathbb{R}^2$, there exists a point X such that $\alpha(X) = Y$.*
(3) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called bijection *(or* permutation*, or* transformation*) if it is one-to-one and onto.*

Definition 3.2 (Composition)**.** *Let α, β be two maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. The composition $\alpha\beta = \alpha \circ \beta$ is the map from $\mathbb{R}^2$ to itself, defined by*

$$\alpha\beta(X) = \alpha \circ \beta(X) = \alpha(\beta(X)), \quad X \in \mathbb{R}^2.$$

Definition 3.3 (Inverse)**.** *Let α be a bijection map from $\mathbb{R}^2$ to itself. Then, the inverse map $\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the map satisfies $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \iota = \mathrm{Id}$.*

Definition 3.4 (Image of maps)**.** *Let α be a map from $\mathbb{R}^2$ to itself and S be a subset of $\mathbb{R}^2$. The image of S under α is defined by*

$$\alpha(S) = \{\alpha(X) : X \in S\}.$$

Definition 3.5 (Translations)**.** *Let $A \in \mathbb{R}^2$. The translation by A, denoted by $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$, is defined by*

$$\tau_A(X) = X + A, \quad X \in \mathbb{R}^2.$$

Example 3.6. *(1) Let $\mathrm{Id} = \iota : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\iota(X) = X$. This map is called the identity.*
(2) Let $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mu(X) = 2X$.
(3) Let $A \in \mathbb{R}^2$. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\gamma(X) = A$.

Proposition 3.7. *Let $A, B \in \mathbb{R}^2$.*

(1) The translation τ_A is one-to-one and onto.
(2) $\tau_A\tau_B = \tau_{A+B}$.
(3) $\tau_A^{-1} = \tau_{-A}$.
(4) τ_A maps a line ℓ to a line $\tau_A(\ell)$, and $\tau_A(\ell) \mathbin{/\!/} \ell$.
(5) For fixed $B, C \in \mathbb{R}^2$, there exists a unique A such that $\tau_A(B) = C$.

REFERENCES

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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