MATH 403 LECTURE NOTE
WEEK 1

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1. Operations of vectors (Sec 1.1-3)

The set of all real numbers is denoted by $\mathbb{R}$. For any $a_1, a_2 \in \mathbb{R}$, a pair (a_1, a_2) is called a vector in the plane. The set of all vectors in the plane is denoted by $\mathbb{R}^2$. In this course, vectors will be denoted by upper case letters, that is, $A = (a_1, a_2), B = (b_1, b_2)$, etc.

There are two geometric interpretations for vectors. We identify a vector $A = (a_1, a_2)$ with a point (a_1, a_2) in the plane. We use $O = (0, 0)$ for the origin. When we use vectors for points, we will just use upper case letters.

On the other hand, we use a vector A to describe an arrow from the origin to A. When vectors are used for arrows, we denote by

$$\overrightarrow{OA} = (a_1, a_2).$$

Definition 1.1. *We say two vectors $A = (a_1, a_2), B = (b_1, b_2)$ are equal, denoted by $A = B$, if and only if $a_1 = b_1$ and $a_2 = b_2$.*

Definition 1.2. *For any vectors $A = (a_1, a_2), B = (b_1, b_2)$ and a real number $r \in \mathbb{R}$, we define*

$$A + B = (a_1 + b_1, a_2 + b_2),$$
$$rA = (ra_1, ra_2).$$

Proposition 1.3. *For any vectors $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$ and real numbers $r, s \in \mathbb{R}$, we have*

- (A1) $A + B = B + A$ *(commutativity)*
- (A2) $A + (B + C) = (A + B) + C$ *(associativity)*
- (A3) $O + A = A = A + O$ *(identity)*
- (A4) $A + (-A) = O$ *(inverse)*
- (M1) $(r + s)A = rA + sA$
- (M2) $r(A + B) = rA + rB$
- (M3) $r(sA) = (rs)A$
- (M4) $1A = A$

Proof. (A1): By definition of the addition, we have $A+B = (a_1+b_1, a_2+b_2)$ and $B+A = (b_1+a_1, b_2+a_2)$. By the commutativity of real numbers, we know $a_1 + b_1 = b_1 + a_1$ and $a_2 + b_2 = b_2 + a_2$. Since each components are equal, we conclude that $A + B = B + A$.

The rest are exercise. ■

We consider the vector $\overrightarrow{OA}$. Then translate $\overrightarrow{OB}$ to the end of $\overrightarrow{OA}$. The sum $A + B$ is a vector from the starting point of $\overrightarrow{OA}$ to the end point of the translated vector $\overrightarrow{OB}$. The vector $-A$ can be thought of as an arrow from O to $(-a_1, -a_2)$. Thus, $-A$ is the opposite direction to A. All scalar multiples of A, rA, lie on the line through from the origin to A. Combining the addition and the scalar multiplication, we define the subtraction $A - B$ by $A + (-B)$. What is the geometric meaning of $A - B$? Consider $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then an arrow from the point B to the point A corresponds to $A - B$. In other words, $\overrightarrow{AB} = A - B$. Note that $A = \overrightarrow{OA} = A - O$.

When we identify a vector with an arrow, we will disregard its location, only its direction and length matter. So we say two arrow are equal if they are identical up to translation. In terms of vectors, $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$.

Definition 1.4. *The vectors $E_1 = (1, 0)$ and $E_2 = (0, 1)$ are called the standard basis vectors.*

They are called basis because every vector in the plane can be *uniquely* written in terms of E_1 and E_2. Indeed,

$$A = (a_1, a_2) = (a_1, 0) + (0, a_2) = a_1E_1 + a_2E_2.$$

(Is this representation unique? Exercise.)

In general,

Proposition 1.5. *Suppose B and C are distinct nonzero vectors in the plane such that the lines ℓ_{OB} (through O and B) and ℓ_{OC} (through O and C) are distinct. Then, every vector A has a unique representation*

$$A = rB + sC$$

for some $r, s \in \mathbb{R}$.

Proof. Consider a system of linear equations

$$\begin{cases} a_1 = b_1r + c_1s, \\ a_2 = b_2r + c_2s \end{cases}$$

Then, the system has a unique solution if and only if $b_1c_2 - b_2c_1 \neq 0$. The latter condition is satisfied because of the assumption $\ell_{OB} \neq \ell_{OC}$. Suppose $b_1c_2 = b_2c_1$. Since B and C are nonzero, either b_1 and c_1 are nonzero, or b_2 and c_2 are nonzero. (why?) Without loss of generality, we assume that $b_1 \neq 0$ and $c_1 \neq 0$. Then, we get $b_2/b_1 = c_2/c_1$, which implies that $\ell_{OB} = \ell_{OC}$. This is a contradiction. Thus, $b_1c_2 \neq b_2c_1$. ■

2. Equation of a line (Sec 1.4-5)

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two distinct points. We want to describe the line through A and B in terms of vectors. Suppose a point P lies on the line ℓ_{AB}. Then, the vector $\overrightarrow{PA}$ and $\overrightarrow{AB}$ have the same direction, with possibly different length. In other words, $\overrightarrow{PA}$ is a scalar multiple of $\overrightarrow{AB}$. Thus,

$$\overrightarrow{PA} = A - P = t(B - A) = t\overrightarrow{AB}$$

for some $t \in \mathbb{R}$.

Definition 2.1. *The line ℓ_{AB} through A and B is the set of all points P that satisfy $\overrightarrow{PA} = t\overrightarrow{AB}$ for some $t \in \mathbb{R}$, that is,*

$$\ell_{AB} = \{P \in \mathbb{R}^2 : \overrightarrow{PA} = t\overrightarrow{AB} \quad \text{for some } t \in \mathbb{R}\}.$$

Theorem 2.2. *Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two distinct points. Then every point P lies on the line ℓ_{AB} through A and B if and only if there exist $a, b \in \mathbb{R}$ such that $a + b = 1$ and*

$$P = aA + bB.$$

Furthermore, this representation is unique: if $aA + bB = cA + dB$ with $a + b = c + d = 1$, then $a = c$ and $b = d$.

Proof. By definition, if $P \in \ell_{AB}$, then there exists a real number $t \in \mathbb{R}$ such that $\overrightarrow{PA} = t\overrightarrow{AB}$. From $A - P = t(B - A)$, we have $P = (1 + t)A - tB$. Let $a = 1 + t$ and $b = -t$, then $a + b = 1$.

If $P = aA + bB$ with $a, b \in \mathbb{R}$ and $a + b = 1$, then with $t = -b$ we have $\overrightarrow{PA} = t\overrightarrow{AB}$. Thus, $P \in \ell_{AB}$.

To show the uniqueness, we assume that there are $a, b, c, d \in \mathbb{R}$ such that $a + b = c + d = 1$ and

$$P = aA + bB = cA + dB.$$

Thus, $(a - c)A = (d - b)B$. Since $d - b = (1 - c) - (1 - a) = a - c$, we obtain$(a - c)a_1 = (a - c)b_1$ and $(a - c)a_2 = (a - c)b_2$. Thus,

$$(a - c)(a_1 - b_1) = 0 = (a - c)(a_2 - b_2).$$

Since $A \neq B$, either $a_1 - b_1 \neq 0$ or $a_2 - b_2 \neq 0$. Thus, $a - c = 0$ as desired. ■

Definition 2.3. *The* midpoint *of points A and B (or of the segment AB) is a point M that satisfies $\overrightarrow{AM} = \overrightarrow{MB}$.*

Exercise 3. *Show that $\frac{1}{2}(A + B)$ is the midpoint of A and B.*

Definition 3.1. *Let $A, B, C, D \in \mathbb{R}^2$ be distinct. Then the lines ℓ_{AB} and ℓ_{CD} are* parallel, *denoted by $\ell_{AB} \mathbin{/\!/} \ell_{CD}$, if $D - C = t(B - A)$ for some nonzero $t \in \mathbb{R}$.*

Exercise 4. *If lines ℓ_{AB} and ℓ_{CD} are parallel and $C \notin \ell_{AB}$, then $\ell_{AB} \cap \ell_{CD} = \emptyset$.*

5. Parallelograms (Sec 1.6)

Let A, B, C, D be distinct points in the plane. A quadrilateral $ABCD$ (in order) is defined by joining A and B, B and C, C and D and D and A. Our question is when a quadrilaterel $ABCD$ forms a parallelogram. To be a parallelogram, the sides $\overline{AB}$ and $\overline{DC}$ are parallel and have the same length. Equivalently, the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal. Thus,

$$\overrightarrow{AB} = B - A = C - D = \overrightarrow{DC},$$
$$A + C = B + D.$$

This gives us a definition.

Definition 5.1. *Four points A, B, C, D define a parallelogram $ABCD$ if $A + C = B + D$.*

Remark 5.2. *We do not assume that A, B, C, D are distinct. Thus, our definition generalizes parallelograms. For instance, if $A = B$, then $C = D$ by the equation. Thus, the parallelogram $ABCD$ is just a line segment $\overline{AD} = \overline{BC}$.*

Proposition 5.3. *A quadrilateral is a parallelogram if and only if the diagonals bisect each other.*

Proof. Suppose a quadrilateral $ABCD$ is a parallelogram. From the defining equation, we have $\frac{1}{2}(A+C) = \frac{1}{2}(B+D)$. Since $A+C$ and $B+D$ are the diagonals of $ABCD$, we see that $\frac{1}{2}(A+C)$ and $\frac{1}{2}(B+D)$ are the midpoints of them. Thus, the diagonals bisect each other.

Suppose the diagonals of a quadrilateral $ABCD$ bisect each other. Then, vectors A, B, C, D should satisfy $\frac{1}{2}(A+C) = \frac{1}{2}(B+D)$, which implies the defining equation. Thus, $ABCD$ is a parallelogram. ∎

6. Centroid (Sec 1.7-8)

Definition 6.1. *Consider a triagle $\triangle ABC$. A median is a line joining a vertex to the midpoint of the opposite side.*

Theorem 6.2. *The medians of a triagle are concurrent. That is, the medians intersect in one point.*

Proof. By translation, we assume that $A = O$. We also assume that B and C are not on the same line. Otherwise, ABC does not form a triangle. Let A' be the midpoint of B and C, and B' be the midpoint of C and A. Consider $P \in \ell_{AA'}$ and $Q \in \ell_{BB'}$, then

$$P = P(t) = (1-t)A + tA' = (1-t)A + \frac{t}{2}B + \frac{t}{2}C = \frac{t}{2}B + \frac{t}{2}C,$$
$$Q = Q(s) = (1-s)B + sB' = (1-s)B + \frac{s}{2}C + \frac{s}{2}A = (1-s)B + \frac{s}{2}C.$$

Thus, if $P = Q$, then

$$(1 - s - \frac{t}{2})B = \frac{t-s}{2}C.$$

Since B and C are not on the same line, the coefficients should be zero. Thus, $t = s = \frac{2}{3}$. Thus, the intersection of the two medians is

$$\ell_{AA'} \cap \ell_{BB'} = \left\{\frac{1}{3}(A+B+C)\right\}.$$

Since the point is on the other median $\ell_{CC'}$, where $C' = \frac{1}{2}(A+B)$ (why?), the proof is complete. ∎

Definition 6.3. *The* centroid *of a triangle is a unique point in the intersection of three medians. According to the proof of the previous theorem, the centroid can be written as*

$$G = \frac{1}{3}(A+B+C).$$

In general,

Definition 6.4. *The centroid G of $\{A_1, A_2, \cdots, A_n\} \subset \mathbb{R}^2$ is defined by*

$$G = \frac{1}{n}(A_1 + A_2 + \cdots + A_n).$$

Note that G is the midpoint if $n = 2$ and the centroid of a triangle $\triangle ABC$ if $n = 3$.

Theorem 6.5. *Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$. Let G_1 be the centroid of A_2, A_3, A_4, G_2 be the centroid of A_1, A_3, A_4, and so on. Let G be the centroid of A_1, A_2, A_3, A_4. Then,*

$$G \in \ell_{A_1G_1} \cap \ell_{A_2G_2} \cap \ell_{A_3G_3} \cap \ell_{A_4G_4}.$$

Proof. By symmetry, it suffices to show that $G \in \ell_{A_1G_1}$. Every point P on the line $\ell_{A_1G_1}$ can be written as

$$P = (1-t)A_1 + tG_1 = (1-t)A_1 + \frac{t}{3}A_2 + \frac{t}{3}A_3 + \frac{t}{3}A_4$$

for some $t \in \mathbb{R}$. If $t = \frac{1}{4}$, then one can see that $G = P$. Thus, $G \in \ell_{A_1G_1}$. ■

Remark 6.6. *Do the lines $\ell_{A_iG_i}$, $i = 1, 2, 3, 4$ intersect in a single point? In other words,*

$$\ell_{A_1G_1} \cap \ell_{A_2G_2} \cap \ell_{A_3G_3} \cap \ell_{A_4G_4} = \{G\}?$$

Proposition 6.7. *Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$. Let M_1 be the midpoint of A_1, A_2 and M_2 be the midpoint of A_3, A_4. Let G be the centroid of A_1, A_2, A_3, A_4. Then, $G \in \ell_{M_1M_2}$.*

Proof. Every point P on $\ell_{M_1M_2}$ can be written as

$$P = (1-t)M_1 + tM_2 = \frac{1-t}{2}A_1 + \frac{1-t}{2}A_2 + \frac{t}{2}A_3 + \frac{t}{2}A_4$$

for some $t \in \mathbb{R}$. If $t = \frac{1}{2}$, then $P = G$. ■

Proposition 6.8. *Let $n, p, q \in \mathbb{N}$ with $p + q = n$. Let $A_1, A_2, \cdots, A_n \in \mathbb{R}^2$. Let U be the centroid of $A_1, \cdots, A_p$ and V be the centroid of $A_{p+1}, \cdots, A_n$. Let G be the centroid of $A_1, A_2, \cdots, A_n$. Then, $G \in \ell_{UV}$.*

Proof. The result follows from

$$G = \frac{p}{n}U + \frac{q}{n}V.$$

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References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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