MATH 403 LECTURE NOTE
WEEK 6

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1. Translations and Central dilatations (Sec 2.1–2)

Consider an assignment α from the set of points of the plane to itself. We call α a *map* or a *correspondence*. If α assigns a point X to Y, we use the notation $\alpha(X) = Y$. We say two maps α and β are *equal* if $\alpha(X) = \beta(X)$ for all $X \in \mathbb{R}^2$.

Definition 1.1. *(1) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called* one-to-one *if $\alpha(X) = \alpha(X')$ implies $X = X'$.*

(2) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called onto *if for every $Y \in \mathbb{R}^2$, there exists a point X such that $\alpha(X) = Y$.*

(3) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called bijection *(or* permutation*, or* transformation*) if it is one-to-one and onto.*

Definition 1.2 (Composition)**.** *Let α, β be two maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. The composition $\alpha\beta = \alpha \circ \beta$ is the map from $\mathbb{R}^2$ to itself, defined by*

$$\alpha\beta(X) = \alpha \circ \beta(X) = \alpha(\beta(X)), \quad X \in \mathbb{R}^2.$$

Definition 1.3 (Inverse)**.** *Let α be a bijection map from $\mathbb{R}^2$ to itself. Then, the inverse map $\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the map satisfies $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \iota = \mathrm{Id}$.*

Remark 1.4. *Note that if $\alpha \circ \beta$ is one-to-one, then β is one-to-one. If $\alpha \circ \beta$ is onto, then α is onto.*

Definition 1.5 (Image of maps)**.** *Let α be a map from $\mathbb{R}^2$ to itself and S be a subset of $\mathbb{R}^2$. The image of S under α is defined by*

$$\alpha(S) = \{\alpha(X) : X \in S\}.$$

Definition 1.6 (Translations)**.** *Let $A \in \mathbb{R}^2$. The translation by A, denoted by $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$, is defined by*

$$\tau_A(X) = X + A, \quad X \in \mathbb{R}^2.$$

Proposition 1.7. *Let $A, B \in \mathbb{R}^2$.*

(1) The translation τ_A is one-to-one and onto.
(2) $\tau_A\tau_B = \tau_{A+B}$.
(3) $\tau_A^{-1} = \tau_{-A}$.
(4) τ_A maps a line ℓ to a line $\tau_A(\ell)$, and $\tau_A(\ell) /\!/ \ell$.
(5) For fixed $B, C \in \mathbb{R}^2$, there exists a unique A such that $\tau_A(B) = C$.

Definition 1.8. *A fixed point of a map α is a point $X \in \mathbb{R}^2$ such that $\alpha(X) = X$.*

Definition 1.9 (Central dilatations)**.** *Let r be a nonzero number. The central dilatation with center O and dilatation factor r is the map $\delta_r : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\delta_r(X) = rX, \quad X \in \mathbb{R}^2.$$

Proposition 1.10. *Let $r, s \in \mathbb{R} \setminus \{0\}$ and $A \in \mathbb{R}^2$.*

(1) The map δ_r is one-to-one and onto.
(2) $\delta_r \circ \delta_s = \delta_{rs}$.
(3) $(\delta_r)^{-1} = \delta_{1/r}$.
(4) $\delta_r \circ \tau_A = \tau_{rA} \circ \delta_r$. In particular, $\tau_{rA} = \delta_r \circ \tau_A \circ (\delta_r)^{-1}$.

Definition 1.11. *Let $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection,and $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ a map. The conjugate $\overline{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ of α by μ is defined by*

$$\overline{\alpha} = \mu \circ \alpha \circ \mu^{-1}.$$

Thus, τ_{rA} is the conjugate of the translation τ_A by the central dilatation δ_r.

Definition 1.12. *Let $C \in \mathbb{R}^2$ and $r \in \mathbb{R}$ with $r \neq 0$. The dilatation with center C and dilatation factor r is the map $\delta_{C,r} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\delta_{C,r}(X) = C + r(X - C) = (1 - r)C + rX, \quad X \in \mathbb{R}^2.$$

Note that $\delta_{C,r}$ is a bijection (exercise) and $\delta_{O,r} = \delta_r$.

Proposition 1.13. *Let $A, C \in \mathbb{R}^2$ and $r, s \in \mathbb{R} \setminus \{0\}$.*

(1) $\delta_{C,r} \circ \delta_{C,s} = \delta_{C,rs}$, $\delta_{C,1} = \mathrm{Id}$, and $(\delta_{C,r})^{-1} = \delta_{C,1/r}$.
(2) The map $\delta_{A+C,r}$ is the conjugate of $\delta_{A,r}$ by τ_C. That is, $\delta_{A+C,r} = \tau_C \circ \delta_{A,r} \circ (\tau_C)^{-1}$.
(3) C is the fixed point of $\delta_{C,r}$, that is, $\delta_{C,r}(C) = C$. The point C is the only fixed point if and only if $r \neq 1$.
(4) $\delta_{C,r}$ maps a line to a parallel line.
(5) If A, B, C are distinct and collinear, then there exists a unique central dilatation $\delta_{C,r}$ with $\delta_{C,r}(A) = B$.

Proof. (1) For $X \in \mathbb{R}^2$, we have

$$\begin{aligned}\delta_{C,r} \circ \delta_{C,s}(X) &= \delta_{C,r}((1 - s)C + sX) = (1 - r)C + r((1 - s)C + sX) \\ &= (1 - rs)C + rsX = \delta_{C,rs}(X)\end{aligned}$$

and

$$\delta_{C,1}(X) = (1 - 1)C + 1X = X.$$

Since $\delta_{C,r} \circ \delta_{C,1/r} = \mathrm{Id} = \delta_{C,1/r} \circ \gamma_{C,r}$, the inverse of $\delta_{C,r}$ is $\delta_{C,1/r}$.

(2) HW.

(3) $\delta_{C,r}(C) = (1 - r)C + rC = C$. If $r = 1$, then $\delta_{C,r} = \mathrm{Id}$ so every point is a fixed point. Suppose $r \neq 1$ and $X \neq C$, then

$$\delta_{C,r}(X) - X = (1 - r)(C - X) \neq O.$$

(4) HW.

(5) It suffices to find r such that $\delta_{C,r}(A) = B$. Since A, B, C are collinear, in particular we have $B \in \ell_{AC}$. Thus, there exists $r \in \mathbb{R}$ such that $B = (1 - r)C + rA$. Then,

$$\delta_{C,r}(A) = (1 - r)C + rA = B$$

as desired.

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Theorem 1.14. *Let A, B, C form a triangle and A', B', C' be the midpoints of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Let Q be any point in $\mathbb{R}^2$. Let ℓ_1 be the line through A', parallel to ℓ_{AQ}, ℓ_2 the line through B', parallel to ℓ_{BQ}, and ℓ_3 the line through C', parallel to ℓ_{CQ}.*

(1) The three lines are concourrent in a point P, and
(2) the centroid G lies on ℓ_{PQ} and

$$\frac{G - P}{G - Q} = -\frac{1}{2}.$$

Proof. Let G be the centroid of $\triangle ABC$. Consider a map $\alpha = \delta_{G,-1/2}$.One can see that

$$\alpha(A) = \left(1 - \left(-\frac{1}{2}\right)\right) G - \frac{1}{2}A = \frac{1}{2}(B + C) = A'.$$

Similarly, we have $\alpha(B) = B'$ and $\alpha(C) = C'$. Let $P = \alpha(Q)$. Since α maps a line to a parallel line, we know that $\alpha(\ell_{AQ}) = \ell_1 = \ell_{A'P}$, $\alpha(\ell_{BQ}) = \ell_2 = \ell_{B'P}$, and $\alpha(\ell_{CQ}) = \ell_3 = \ell_{C'P}$. In particular, we have $P \in \ell_1 \cap \ell_2 \cap \ell_3$.

Since $\alpha(Q) = P$, we have

$$\begin{aligned}P &= \frac{3}{2}G - \frac{1}{2}Q, \\ G - P &= -\frac{1}{2}G + \frac{1}{2}Q = -\frac{1}{2}(G - Q).\end{aligned}$$

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2. Central Reflections

Definition 2.1. *The central reflection in C is the bijection $\sigma_C = \delta_{C,-1}$, that is,*

$$\sigma_C(X) = 2C - X.$$

Note that C is the midpoint of X and $\sigma_C(X)$. A special case is when $C = O$, $\sigma_O(X) = -X$.

Proposition 2.2. *Let $A, B, C \in \mathbb{R}^2$.*

(1) $\sigma_C^2 = \sigma_C \circ \sigma_C = \mathrm{Id}$.
(2) C is the unique fixed point of σ_C.
(3) The composition of two central reflections is a translation.
(4) The composition $\sigma_C\sigma_B\sigma_A$ is the central reflection σ_D where D is the fourth vertex of the parallelogram $ABCD$.

Proof. (1) For $X \in \mathbb{R}^2$, $\sigma_C^2(X) = \sigma_C(2C - X) = 2C - (2C - X) = X$.

(2) One sees $\sigma_C(C) = 2C - C = C$. If $X \neq C$, then

$$\sigma_C(X) - X = 2C - X - X = 2(C - X) \neq O.$$

(3) For $A, B, X \in \mathbb{R}^2$, we have

$$\sigma_A\sigma_B(X) = \sigma_A(2B - X) = 2A - (2B - X) = 2(A - B) + X = \tau_{2(A-B)}(X).$$

(4) Let $D = C - B + A$, then A, B, C, D defines a parallelogram and

$$\sigma_C\sigma_B\sigma_A(X) = \sigma_C(2(B - A) + X) = 2(C - B + A) - X = \sigma_D(X).$$

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References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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