MATH 403 FALL 2021: HOMEWORK 9 SOLUTION

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1. Exercise 3.14 (Hint: Use the orthogonal projection of Y to X.)

Solution: Let $Z = X/|X|$ and $W = Y/|Y|$, then

$$\begin{aligned}|\triangle OXY| &= \frac{1}{2}|X||Y - \mathrm{Proj}_X Y| \\ &= \frac{1}{2}|X|\left|Y - \frac{X\cdot Y}{|X|^2}X\right| \\ &= \frac{1}{2}|X||Y|\left|\frac{Y}{|Y|} - \frac{X\cdot Y}{|X||Y|}\frac{X}{|X|}\right| \\ &= \frac{1}{2}|X||Y|\,|W-(Z\cdot W)Z|\,.\end{aligned}$$

Since $|W-(Z\cdot W)Z|^2 = 1-(Z\cdot W)^2$, we have

$$\begin{aligned}|\triangle OXY| &= \frac{1}{2}|X||Y|\,|W-(Z\cdot W)Z| \\ &= \frac{1}{2}|X||Y|\sqrt{1-(Z\cdot W)^2} \\ &= \frac{1}{2}\sqrt{|X|^2|Y|^2-(X\cdot Y)^2}.\end{aligned}$$

2. Exercise 3.16

Solution: Direct computation yields

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} &= \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ &= \frac{1}{4}\sqrt{((a+b)^2-c^2)(c^2-(a-b)^2)} \\ &= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \\ &= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \\ &= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}.\end{aligned}$$

Applying

$$c^2 = |X-Y|^2 = |X|^2+|Y|^2-2|X||Y|\cos\theta = a^2+b^2-2ab\cos\theta,$$

we get

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} &= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}\\ &= \frac{1}{2}ab\sqrt{1-\cos^2\theta}\\ &= \frac{1}{2}|X||Y||\sin\theta|.\end{aligned}$$

3. Exercise 3.17

Solution:

$$\begin{aligned}&X\cdot N = x_1n_1+x_2n_2 = n_1p_1+n_2p_2 = N\cdot P,\\ &n_1(x_1-p_1)+n_2(x_2-p_2)=0.\end{aligned}$$

4. Exercise 3.18

Solution: If N is perpendicular to N' and $N, N' \neq O$, then ℓ is perpendicular to ℓ'. Let $\ell = \ell_{AB}$ and $\ell' = \ell_{CD}$, then $N\cdot(A-B) = N'\cdot(C-D) = 0$. Let $X = A-B$ and $Y = C-D$. Suppose N, N', O are collinear, then $N = rN'$ for some $r\in\mathbb{R}$. Then, $N\cdot N = rN\cdot N' = 0$, which contradicts to the assumption. Thus, there exist a, b such that $X = aN+bN'$. Since $X\cdot N = 0$ and $N\neq O$, we have $a = 0$. Then, $X\cdot Y = bN'\cdot Y = 0$ as desired.

5. For $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, we define

$$d_\infty(X,Y) := \max\{|x_1-y_1|, |x_2-y_2|\}.$$

Show that $d_1(\cdot,\cdot)$ satisfies the following:
(a) $d_\infty(X,Y) = d_\infty(Y,X)$ for any $X, Y\in\mathbb{R}^2$.
(b) $d_\infty(X,Y) \geqslant 0$ for any $X, Y\in\mathbb{R}^2$ and $d_\infty(X,Y)$ equals to zero if and only if $X = Y$.
(c) $d_\infty(X,Z) \leq d_\infty(X,Y) + d_\infty(Y,Z)$ for any $X, Y, Z\in\mathbb{R}^2$.

Solution:
(a) $d_\infty(X,Y) = \max\{|x_1-y_1|, |x_2-y_2|\} = \max\{|y_1-x_1|, |y_2-x_2|\} = d_\infty(Y,X)$.
(b) $d_\infty(X,Y)\geq 0$ is obvious. Suppose $d_\infty(X,Y) = 0$, then $|x_1-y_1| = |x_2-y_2| = 0$, which implies $X = Y$.
(c) Suppose that $d_\infty(X,Z) = |x_1-z_1|$ without loss of generality. Then, we have

$$d_\infty(X,Z) = |x_1-z_1| \leq |x_1-y_1| + |y_1-z_1| \leq d_\infty(X,Y) + d_\infty(Y,Z).$$

6. (Continued) Draw the set $\{X : d_\infty(X,D) = 2\}$ for $D = (2,3)$ in the plane.

Solution: If X satisfies $d_\infty(X,D) = 2$, then

$$|x_1-2| = 2,\quad |x_2-3|\leq 2,\quad \text{or}\quad |x_1-2|\leqslant 2,\quad |x_2-3| = 2.$$

Thus,

$$\begin{cases} x_1 = 4, 1\leq x_2\leqslant 5,\\ x_1 = 0, 1\leq x_2\leqslant 5,\\ x_2 = 5, 0\leq x_1\leqslant 4,\\ x_2 = 1, 0\leq x_1\leqslant 4.\end{cases}$$