MATH 403 LECTURE NOTE
WEEK 5

DAESUNG KIM

1. TRANSLATIONS AND CENTRAL DILATATIONS (SEC 2.1–2)

Consider an assignment α from the set of points of the plane to itself. We call α a *map* or a *correspondence*. If α assigns a point X to Y, we use the notation $\alpha(X) = Y$. We say two maps α and β are *equal* if $\alpha(X) = \beta(X)$ for all $X \in \mathbb{R}^2$.

Definition 1.1. *(1) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called* one-to-one *if $\alpha(X) = \alpha(X')$ implies $X = X'$.*
(2) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called onto *if for every $Y \in \mathbb{R}^2$, there exists a point X such that $\alpha(X) = Y$.*
(3) A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called bijection *(or* permutation*, or* transformation*) if it is one-to-one and onto.*

Definition 1.2 (Composition)**.** *Let α, β be two maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. The composition $\alpha\beta = \alpha \circ \beta$ is the map from $\mathbb{R}^2$ to itself, defined by*

$$\alpha\beta(X) = \alpha \circ \beta(X) = \alpha(\beta(X)), \quad X \in \mathbb{R}^2.$$

Definition 1.3 (Inverse)**.** *Let α be a bijection map from $\mathbb{R}^2$ to itself. Then, the inverse map $\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the map satisfies $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \iota = \mathrm{Id}$.*

Definition 1.4 (Image of maps)**.** *Let α be a map from $\mathbb{R}^2$ to itself and S be a subset of $\mathbb{R}^2$. The image of S under α is defined by*

$$\alpha(S) = \{\alpha(X) : X \in S\}.$$

Definition 1.5 (Translations)**.** *Let $A \in \mathbb{R}^2$. The translation by A, denoted by $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$, is defined by*

$$\tau_A(X) = X + A, \quad X \in \mathbb{R}^2.$$

Proposition 1.6. *Let $A, B \in \mathbb{R}^2$.*
1. *The translation τ_A is one-to-one and onto.*
2. *$\tau_A\tau_B = \tau_{A+B}$.*
3. *$\tau_A^{-1} = \tau_{-A}$.*
4. *τ_A maps a line ℓ to a line $\tau_A(\ell)$, and $\tau_A(\ell) \parallel \ell$.*
5. *For fixed $B, C \in \mathbb{R}^2$, there exists a unique A such that $\tau_A(B) = C$.*

Definition 1.7 (Central dilatations)**.** *Let r be a nonzero number. The central dilatation with center O and dilatation factor r is the map $\delta_r : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\delta_r(X) = rX, \quad X \in \mathbb{R}^2.$$

Proposition 1.8. *Let $r, s \in \mathbb{R} \setminus \{0\}$ and $A \in \mathbb{R}^2$.*
1. *The map δ_r is one-to-one and onto.*
2. *$\delta_r \circ \delta_s = \delta_{rs}$.*
3. *$(\delta_r)^{-1} = \delta_{1/r}$.*
4. *$\delta_r \circ \tau_A = \tau_{rA} \circ \delta_r$. In particular, $\tau_{rA} = \delta_r \circ \tau_A \circ (\delta_r)^{-1}$.*

Definition 1.9. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection,and $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ a map. The conjugate $\overline{\mu} : \mathbb{R}^2 \to \mathbb{R}^2$ of μ by α is defined by*

$$\overline{\mu} = \alpha \circ \mu \circ \alpha^{-1}.$$

Thus, τ_{rA} is the conjugate of the translation τ_A by the central dilatation δ_r.

Definition 1.10. *Let $C \in \mathbb{R}^2$ and $r \in \mathbb{R}$ with $r \neq 0$. The dilatation with center C and dilatation factor r is the map $\delta_{C,r} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\delta_{C,r}(X) = C + r(X - C) = (1 - r)C + rX, \quad X \in \mathbb{R}^2.$$

Note that $\delta_{C,r}$ is a bijection (exercise) and $\delta_{O,r} = \delta_r$.

Proposition 1.11. *Let $A, C \in \mathbb{R}^2$ and $r, s \in \mathbb{R} \setminus \{0\}$.*

(1) $\delta_{C,r} \circ \delta_{C,s} = \delta_{C,rs}$, $\delta_{C,1} = \mathrm{ID}$, and $(\delta_{C,r})^{-1} = \delta_{C,1/r}$.
(2) The map $\delta_{A+C,r}$ is the conjugate of $\delta_{A,r}$ by τ_C. That is, $\delta_{A+C,r} = \tau_C \circ \delta_{A,r} \circ (\tau_C)^{-1}$.
(3) C is the fixed point of $\delta_{C,r}$, that is, $\delta_{C,r}(C) = C$. The point C is the only fixed point if and only if $r \neq 1$.
(4) $\delta_{C,r}$ maps a line to a parallel line.

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu