MATH 403 LECTURE NOTE
WEEK 9

DAESUNG KIM

1. ORTHOCENTERS (SEC. 3.2)

Definition 1.1. *The altitude ℓ_C of a triangle $\triangle ABC$ through C is the line perpendicular to ℓ_{AB} through C. The intersection point H_C between ℓ_C and ℓ_{AB} is called the foot of ℓ_C.*

Theorem 1.2. *The altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter.*

There are three proofs.

First Proof. This proof relies on the following observation.

Lemma 1.3. *For any points $X, A, B, C \in \mathbb{R}^2$, we have*

$$(X - A) \cdot (B - C) + (X - B) \cdot (C - A) + (X - C) \cdot (A - B) = 0.$$

If $X \in \ell_A \cap \ell_B$, then

$$(X - A) \cdot (B - C) = (X - B) \cdot (C - A) = 0.$$

Thus, we have $(X - C) \cdot (A - B) = 0$, which implies that $X \in \ell_C$.

■

Lemma 1.4. *Let ℓ_1, ℓ_2, ℓ_3 be lines. If ℓ_1 is parallel to ℓ_2 and perpendicular to ℓ_3, then ℓ_2 is perpendicular to ℓ_3.*

Lemma 1.5. *Consider a triangle $\triangle ABC$. Let A' be the midpoint of B and C. Let ℓ_n be the perpendicular bisector through A', and ℓ_A the altitude through A. Then, $\delta_{G,-2}$ maps ℓ_n to ℓ_A, where G is the centroid.*

Second Proof. Since $\delta_{G,-2}$ maps each perpendicular bisector to the corresponding altitude and the perpendicular bisectors are concurrent, so are the altitudes. Note that it also maps the concurrence point, the circumcenter, to the concurrence point of the altitudes. ■

Third Proof. Consider the image of $\triangle ABC$ under $\delta_{G,-2}$. Then, the image is also a triangle $\triangle PQR$ and A, B, C are the midpoints of P, Q, R. Since the perpendicular bisectors of $\triangle PQR$ coincide with the altitude of $\triangle ABC$, the proof is complete. ■

REFERENCES

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
E-mail address: daesungk@illinois.edu