Math 403: Euclidean Geometry

Final Exam Solution, Fall 2021

Date: December 10, 2021

1. (14 points) Circle True or False. Do not justify your answer.

   (a) True **FALSE** Let $X, Y \in \mathbb{R}^2$ with $|X| = 4$ and $|Y| = 2$, then $0 \leq X \cdot Y \leq 8$.

   > **Solution:** The Cauchy-Schwarz inequality says that $-|X||Y| \leq X \cdot Y \leq |X||Y|$. In fact, if $X = (4, 0)$ and $Y = (-2, 0)$, then $X \cdot Y = -8$.

   (b) True **FALSE** The composition of two central dilatations is a central dilatation.

   > **Solution:** Let $C, D \in \mathbb{R}^2$ with $C \neq D$, then
   >
   > $$\delta_{C,1/2} \circ \delta_{D,2}(X) = \delta_{C,1/2}(-D + 2X) = \frac{1}{2}(C - D) + X$$
   >
   > is a translation, not a central dilatation.

   (c) **TRUE** False Every involutive isometry has at least one fixed point.

   > **Solution:** Suppose an involutive isometry α has no fixed point. Then, $X \neq \alpha(X)$ for every X. Let M be the midpoint of X and $\alpha(X)$, then
   >
   > $$\alpha(M) = \alpha(\frac{1}{2}(X + \alpha(X))) = \frac{1}{2}\alpha(X) + \frac{1}{2}\alpha(\alpha(X)) = M,$$
   >
   > which is a contradiction.

   (d) True **FALSE** If X, Y, Z are non-collinear, then every point P can be uniquely written as $P = aX + bY + cZ$ with $a, b, c \geq 0$.

   > **Solution:** It holds for all $a, b, c \in \mathbb{R}$ with $a + b + c = 1$.

   (e) True **FALSE** If the lines ℓ and m are parallel, then the composition $\sigma_\ell \circ \sigma_m$ is a central reflection.

   > **Solution:** The composition is a translation.

   (f) **TRUE** False If $|X + 2Y|^2 = |X - 2Y|^2$, then X is perpendicular to Y.

Solution: Since

$$|X + 2Y|^2 = |X|^2 + 4X \cdot Y + 4|Y|^2 = |X|^2 - 4X \cdot Y + 4|Y|^2 = |X - 2Y|^2,$$

we have $X \cdot Y = 0$.

(g) **TRUE** False If $ABCD$ form a rectangle, then there exists a circle $\mathcal{S}$ such that $A, B, C, D \in \mathcal{S}$.

Solution: By Thales theorem, it is true.

2. (12 points) Give definitions of the following.

(a) Centroid

Solution: The *centroid* of a triangle is a unique point in the intersection of three medians. According to the proof of the previous theorem, the centroid can be written as

$$G = \frac{1}{3}(A + B + C).$$

(b) Two lines are parallel

Solution: Let $A, B, C, D \in \mathbb{R}^2$ be distinct. Then the lines ℓ_{AB} and ℓ_{CD} are *parallel* if $D - C = t(B - A)$ for some nonzero $t \in \mathbb{R}$.

(c) Involution

Solution: A map α is an involution if $\alpha \neq \mathrm{Id}$ and $\alpha^2 = \mathrm{Id}$.

(d) A linear map

Solution: A map L is linear if $L(X + Y) = L(X) + L(Y)$ and $L(rX) = rL(X)$ for all $X, Y \in \mathbb{R}^2$ and $r \in \mathbb{R}$.

(e) Orthocenter of a triangle

Solution: The altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter.

(f) Scalar product

Solution: Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. The scalar product of X and Y is defined by

$$X \cdot Y = x_1 y_1 + x_2 y_2.$$

3. (10 points) Write down the statement of one of the following theorems:
   (i) Ceva's theorem,
   (ii) Menelaus's theorem,
   (iii) Nine points circle theorem.

Solution:

(i) Ceva's theorem: Let $A, B, C \in \mathbb{R}^2$ form a triangle. Let A', B', C' be points on the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Assume that A' is distinct from B, C, B' from C, A, and C' from A, B. Then, $\ell_{AA'}, \ell_{BB'}, \ell_{CC'}$ are concurrent if and only if

$$\frac{A'-B}{A'-C} \cdot \frac{B'-C}{B'-A} \cdot \frac{C'-A}{C'-B} = -1.$$

(ii) Menelaus's theorem: Let A, B, C form a triangle. Let $A' \in \ell_{BC}, B' \in \ell_{CA}, C' \in \ell_{AB}$ be distinct from A, B, C. Then, A', B', C' are collinear if and only if

$$\frac{A'-B}{A'-C} \cdot \frac{B'-C}{B'-A} \cdot \frac{C'-A}{C'-B} = 1$$

(iii) Nine points circle theorem: Consider a triangle $\triangle ABC$. Let A', B', C' be the midpoints of A, B, C, H the orthocenter, and

$$A'' = \frac{1}{2}(A+H), \quad B'' = \frac{1}{2}(B+H), \quad C'' = \frac{1}{2}(C+H).$$

Let D, E, F be the feet of the altitudes ℓ_A, ℓ_B, ℓ_C. Then, there exists a circle $\mathcal{S}$ that contains the nine points, $A', B', C', A'', B'', C'', D, E, F$.

4. (a) (5 points) Find an isometry α such that $\alpha \neq \mathrm{Id}$, $\alpha(1,0) = (1,0)$, and $\alpha(-1,0) = (-1,0)$.
   (b) (5 points) Is your example an involution? Justify your answer.

Solution:

(a) Since α has two distinct fixed points, it is either a reflection in the line passing through $(1,0)$ and $(-1,0)$, or the identity. Since it is not the identity and the line is x-axis, α should be $\alpha(x,y) = (x,-y)$.

(b) Since $\alpha^2(x,y) = \alpha(x,-y) = (x,y)$ and it is not the identity, it is an involution.

5. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\alpha(x,y) = \frac{1}{\sqrt{2}}(x-y, x+y)$ for $x, y \in \mathbb{R}$.
   (a) (6 points) Is α an isometry? Justify your answer.
   (b) (6 points) Is α linear? Justify your answer.

Solution:

(a) Yes. For $X = (x, y)$ and $Z = (z, w)$, we have

$$\begin{aligned}|\alpha(X) - \alpha(Z)|^2 &= \frac{1}{2}|((x-z)-(y-w),(x-z)+(y-w))|^2 \\ &= \frac{1}{2}(2(x-z)^2 + 2(y-w)^2) \\ &= |X - Z|^2.\end{aligned}$$

(b) Yes. Since α is an isometry with $\alpha(0,0) = (0,0)$, it is linear.

6. (10 points) Let α be an isometry and $X \in \mathbb{R}^2$ be such that $X \neq \alpha(X)$. Let ℓ be the perpendicular bisector of X and $\alpha(X)$. Show that every fixed point of α lies on the line ℓ.

Solution: Let P be a fixed point. Then,

$$|X - P| = |\alpha(X) - \alpha(P)| = |\alpha(X) - P|,$$

which implies that P is on the perpendicular bisector.

7. (10 points) Let $X, Y, Z \in \mathbb{R}^2$, $a, b, c \in \mathbb{R}$, $a + b + c = 1$, and α be an isometry. Show that

$$\alpha(aX + bY + cZ) = a\alpha(X) + b\alpha(Y) + c\alpha(Z).$$

(Hint: You may use the fact that every isometry can be written as the composition of a translation and a linear isometry.)

Solution: We already know that $\alpha = \tau_R \circ L$ where $R \in \mathbb{R}^2$ and L is a linear isometry. Then,

$$\begin{aligned}\alpha(aX + bY + cZ) &= \tau_R \circ L(aX + bY + cZ) \\ &= \tau_R(aL(X) + bL(Y) + cL(Z)) \\ &= R + aL(X) + bL(Y) + cL(Z) \\ &= aR + bR + cR + aL(X) + bL(Y) + cL(Z) \\ &= a(R + L(X)) + b(R + L(Y)) + c(R + L(Z)) \\ &= a\alpha(X) + b\alpha(Y) + c\alpha(Z).\end{aligned}$$

8. (a) (6 points) Let $P \in \mathbb{R}^2$ and $r \in \mathbb{R}$ with $r \neq 0$. Show that the central dilatation $\delta_{P,r}$ preserves the centroid of three points. (That is, if G is the centroid of A, B, C, then $\delta_{P,r}(G)$ is the centroid of $\delta_{P,r}(A), \delta_{P,r}(B), \delta_{P,r}(C)$.)

(b) (6 points) Let α be an isometry. Does α preserve the centroid of three points? Justify your answer.

Solution:

(a) If follows that

$$\begin{aligned}\delta_{P,r}(G) &= (1-r)P + rG \\ &= (1-r)\left(\frac{1}{3}P + \frac{1}{3}P + \frac{1}{3}P\right) + r\left(\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C\right) \\ &= \frac{1}{3}((1-r)P + rA) + \frac{1}{3}((1-r)P + rB) + \frac{1}{3}((1-r)P + rC) \\ &= \frac{1}{3}\delta_{P,r}(A) + \frac{1}{3}\delta_{P,r}(B) + \frac{1}{3}\delta_{P,r}(C).\end{aligned}$$

(b) Yes. By Problem 7, we have

$$\begin{aligned}\alpha(G) &= \alpha\left(\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C\right) \\ &= \frac{1}{3}\alpha(A) + \frac{1}{3}\alpha(B) + \frac{1}{3}\alpha(C).\end{aligned}$$

9. (10 points) Show that two distinct parallel lines do not intersect.

(Hint: Prove it by contradiction: suppose there exists a point P on the intersection of two lines.)

Solution: Suppose ℓ and m are two distinct parallel lines. Assume that there exists $P \in \ell \cap m$. Since these two lines are distinct, there exist $A, B \in \mathbb{R}^2$ such that $A \in \ell$, $A \notin m$, $B \notin \ell$, and $B \in m$. Since the lines are parallel, there exists $t \in \mathbb{R} \setminus \{0\}$ such that $A - P = t(B - P)$. Thus, $A = (1-t)P + tB \in m$, which is a contradiction.