MATH 403 LECTURE NOTE
WEEK 12

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1. Definition of Isometries (Sec. 4.1)

Definition 1.1. *An isometry is a distance preserving map. That is, a map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if and only if*

$$d(\alpha(X), \alpha(Y)) = d(X, Y) \quad \text{for all } X, Y.$$

Example 1.2. *Translations are isometries. Indeed, let $\alpha = \tau_A$ for $A \in \mathbb{R}^2$, then*

$$|\alpha(X) - \alpha(Y)| = |(X + A) - (Y + A)| = |X - Y|$$

for all X, Y.

Example 1.3. *Central dilatations may not be isometries. Indeed, let $\alpha = \delta_{A,r}$ for $A \in \mathbb{R}^2$ and $r \in \mathbb{R} \setminus \{0\}$, then*

$$|\alpha(X) - \alpha(Y)| = |(rX + (1 - r)A) - (rY + (1 - r)A)| = |r||X - Y|.$$

Thus, α is an isometry if and only if $|r| = 1$. If $r = 1$, then α is the identity. If $r = -1$, then α is the central reflection.

Example 1.4. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$. The following are isometries.*

(1) $\alpha(x_1, x_2) = (x_2, x_1)$
(2) $\alpha(x_1, x_2) = (-x_2, x_1)$
(3) $\alpha(x_1, x_2) = (-x_1, x_2)$
(4) $\alpha(x_1, x_2) = (x_1, -x_2)$

2. Linear Isometries (Sec. 4.1)

Definition 2.1. *A map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear map if*

(1) $\alpha(X + Y) = \alpha(X) + \alpha(Y)$, *and*
(2) $\alpha(rX) = r\alpha(X)$,

for all $X, Y \in \mathbb{R}^2$ and $r \in \mathbb{R}$.

Example 2.2. *The map $\alpha(X) = 2X$ is a linear map. In general, let*

$$\alpha(x_1, x_2) = XM = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (ax_1 + bx_2, cx_1 + dx_2).$$

Then, one can see this is a linear map. In general, this map is not an isometry. Under what condition on the matrix M, is it an isometry? (Exercise)

Example 2.3. *The map $\alpha(x_1, x_2) = (x_1 + 1, x_2 + 3)$ is not a linear map. In fact, if $\alpha = \tau_A$ where $A \neq O$, then α is not linear.*

Remark 2.4. *Note that if α is linear, then $\alpha(O) = O$. Also, it maps a line to a line. Indeed, if ℓ_{AB} is a line, then $X \in \ell_{AB}$ can be written as $X = (1 - t)A + tB$. Then,*

$$\alpha(X) = (1 - t)\alpha(A) + t\alpha(B),$$

which implies $\alpha(X) \in \ell_{\alpha(A)\alpha(B)}$.

Theorem 2.5. *If α is an isometry with $\alpha(O) = O$, then it is linear. In fact, every isometry can be written as $\alpha = L \circ \tau$ where L is linear and τ is a translation.*

Proof. Note that $|\alpha(X)| = |\alpha(X) - \alpha(O)| = |X - O| = |X|$ for all X. Then,

$$0 = |\alpha(X) - \alpha(Y)|^2 - |X - Y|^2 = 2(X \cdot Y - \alpha(X) \cdot \alpha(Y)).$$

Thus,

$$\begin{aligned}
&|\alpha(X+Y) - \alpha(X) - \alpha(Y)|^2 \\
&= |\alpha(X+Y)|^2 + |\alpha(X)|^2 + |\alpha(Y)|^2 - 2(\alpha(X+Y)\cdot\alpha(X) - \alpha(X)\cdot\alpha(Y) + \alpha(X+Y)\cdot\alpha(Y)) \\
&= |X+Y|^2 + |X|^2 + |Y|^2 - 2((X+Y)\cdot X - X\cdot Y + (X+Y)\cdot Y) \\
&= 2|X|^2 + 2|Y|^2 + 2X\cdot Y - 2(|X|^2 + X\cdot Y + |Y|^2) \\
&= 0,
\end{aligned}$$

which implies $\alpha(X+Y) = \alpha(X) + \alpha(Y)$. For $r \in \mathbb{R}$ and $X \in \mathbb{R}^2$, we have

$$\begin{aligned}
|\alpha(rX) - r\alpha(X)|^2 &= |\alpha(rX)|^2 - 2r\alpha(rX)\cdot\alpha(X) + r^2|\alpha(X)|^2 \\
&= |rX|^2 - 2r(rX)\cdot X + r^2|X|^2 \\
&= 0.
\end{aligned}$$

Thus, α is linear. ■

3. Group of Isometries (Sec. 4.1)

Theorem 3.1. *The set of all isometries is a group. In fact, we have the following.*

(1) The composition of two isometries is an isometry.
(2) Every isometry is a bijection.
(3) The inverse of an isometry is an isometry.

Proof. Let α, β be isometries.

(1) It follows from

$$|\alpha(\beta(X)) - \alpha(\beta(Y))| = |\beta(X) - \beta(Y)| = |X - Y|.$$

(2) Suppose $\alpha(X) = \alpha(Y)$. Then, $|\alpha(X) - \alpha(Y)| = |X - Y| = 0$ implies $X = Y$. Thus, α is one-to-one.
Let $E_1 = (1,0)$ and $E_2 = (0,1)$. Let $\beta = \tau_{\alpha(O)} \circ \alpha$, then it suffices to show that β is onto. We cliam that O, $\beta(E_1)$, $\beta(E_2)$ are not collinear. Indeed, if there exists r such that $\beta(E_1) = r\beta(E_2)$, then $\beta(E_1) = \beta(rE_2)$. Since β is one-to-one, $E_1 = rE_2$, which is a contradiction. Thus, any vector $Y \in \mathbb{R}^2$ can be written as $Y = a\beta(E_1) + b\beta(E_2)$. Since β is linear,

$$Y = a\beta(E_1) + b\beta(E_2) = \beta(aE_1 + bE_2).$$

Thus, β is onto and so is α.

(3) For any X, Y, we have

$$|\alpha^{-1}(X) - \alpha^{-1}(Y)| = |\alpha\circ\alpha^{-1}(X) - \alpha\circ\alpha^{-1}(Y)| = |X - Y|.$$

■

Proposition 3.2. *Let ℓ_{AB}, ℓ_{CD} be lines in $\mathbb{R}^2$ with $A \neq B$, $C \neq D$ and α an isometry.*

(1) Every isometry maps a line to a line. Indeed, $\alpha(\ell_{AB}) = \ell_{\alpha(A)\alpha(B)}$.
(2) Every isometry maps two parallel lines to two parallel lines in a sense that if $\ell_{AB} \parallel \ell_{CD}$ then $\ell_{\alpha(A)\alpha(B)} \parallel \ell_{\alpha(C)\alpha(D)}$.
(3) Every isometry maps two perpendicular lines to two perpendicular lines in a sense that if $\ell_{AB} \perp \ell_{CD}$ then $\ell_{\alpha(A)\alpha(B)} \perp \ell_{\alpha(C)\alpha(D)}$.

Proof. Let $\alpha = L \circ \tau_R$ where L is linear and τ_R is a translation.

(1) Since a linear map and a translation map a line to a line and eveery isometry is the composition of a linear map and a translation, the proof is complete.
(2) Let $(A - B) = r(C - D)$. It then follows from

$$\alpha(A) - \alpha(B) = L(A+R) - L(B+R) = L(A-B) = rL(C-D) = r(\alpha(C) - \alpha(D)).$$

(3) Let $(A-B)\cdot(C-D)=0$. Since every linear isometry preserves the scalar product, we have

$$\begin{aligned}(\alpha(A)-\alpha(B))\cdot(\alpha(C)-\alpha(D)) &= (L(A)-L(B))\cdot(L(C)-L(D))\\ &= L((A-B))\cdot L((C-D))\\ &= (A-B)\cdot(C-D)\\ &= 0.\end{aligned}$$

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4. Fixed points of isometries (Sec. 4.2)

Definition 4.1. *Let α be a bijection. We say X is a fixed point of α if $\alpha(X)=X$.*

Proposition 4.2. *Let α be an isometry and X, Y be distinct fixed points of α. Then, every point P on ℓ_{XY} is a fixed point of α.*

Proof. Since $\alpha = L\circ\tau_R$ where L is a linear isometry and τ_R is a translation, for every point $P=(1-t)X+tY\in\ell_{XY}$, we have

$$\begin{aligned}\alpha(P) &= L((1-t)X+tY+R)\\ &= L((1-t)(X+R)+t(Y+R))\\ &= (1-t)L(X+R)+tL(Y+R)\\ &= (1-t)\alpha(X)+t\alpha(Y)\\ &= (1-t)X+tY\\ &= P.\end{aligned}$$

■

Proposition 4.3. *Let α be an isometry and X, Y, Z be distinct, non-collinear, and fixed points of α. Then, $\alpha=\mathrm{Id}$.*

Proof. By assumption, $\ell_{XY}, \ell_{YZ}, \ell_{ZX}$ are distinct. Let $P\in\mathbb{R}^2$. We claim that $\alpha(P)=P$. If $P\in\ell_{XY}\cup\ell_{YZ}\cup\ell_{ZX}$, then $\alpha(P)=P$ by the previous proposition. Suppose P is not on these lines. Consider ℓ_{PX}. If ℓ_{PX} is not parallel to ℓ_{YZ}, then there exists $M\in\ell_{PX}\cap\ell_{YZ}$. Since $M\in\ell_{YZ}$, M is a fixed point. Since $M, X\in\ell_{PX}$ are fixed points, P is also a fixed point. If $\ell_{PX}\mathbin{/\!/}\ell_{YZ}$, then ℓ_{PY} intersects with ℓ_{ZX}. By the same argument, we conclude that P is a fixed point. Thus, α is the identity map. ■

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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