MATH 403 FALL 2021: HOMEWORK 6 SOLUTION

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1. Exercise 2.15

Solution: Suppose $ex = xe = x = e'x = xe'$ for all $x \in G$. Then,

$$e' = ee' = e.$$

2. Exercise 2.16

Solution: Let $x \in G$ and assume that y, z satisfy $xy = yx = e = xz = zx = e$. Then,

$$y = ye = y(xz) = yxz = (yx)z = ez = z.$$

3. Exercise 2.17

Solution: Let $x, y \in G$. Since $xx^{-1} = x^{-1}x = e$, it follows from the uniqueness of the inverse that x is the inverse of x^{-1}, that is $x = (x^{-1})^{-1}$. Also,

$$(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = (xe)x^{-1} = xx^{-1} = e,$$
$$(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = (y^{-1}e)y = y^{-1}y = e,$$

implies that $y^{-1}x^{-1}$ is the inverse of xy.

4. Exercise 2.20

Solution: Since H is a subgroup, it contains the identity $e = x^0$. We claim that $x^m \in H$ for all $m \in \mathbb{N}$. By the assumption $x = x^1 \in H$. Suppose $x^n \in H$, then $x^{n+1} = xx^n \in H$ since the operation is closed in H. By induction on n, we conclude that $x^m \in H$ for all m. Similarly, we can show that $x^{-m} = (x^{-1})^m$ for all $m \in \mathbb{N}$ using induction. Thus, $\langle x \rangle \subseteq H$.

5. Exercise 2.25

Solution: Suppose G be a group with 3 elements. Assume $G = \{e, a, b\}$, e is the identity, and e, a, b are distinct. Suppose $a \cdot a = e$. If $ab = b$ then $a = e$ which contradicts to the assumption. If $ab = e$ then $a = b$ which contradicts to the assumption. If $ab = a$ then $b = e$ which contradicts to the assumption. Thus, $a^2 \neq e$. If $a^2 = a$, then $a = e$ which contradists to the assumption. Thus, the only possibility is that $a^2 = b$. Similarly, $b^2 = a$. Also, $ab = ba = e$. Define $\varphi : G \to \mathbb{Z}_3$ by $\varphi(e) = 0$, $\varphi(a) = 1$, and $\varphi(b) = 2$, then it is a bijection and a homomorphism. Thus, φ is an isomorphism and G is isomorphic to $\mathbb{Z}_3$.