MATH 403 FALL 2021: HOMEWORK 8 SOLUTION

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1. Consider $\triangle ABC$. Let A', B', C' be the midpoints of the sides of $\triangle ABC$. Let H be the orthocenter. Let $A'' = \frac{1}{2}(H + A)$, $B'' = \frac{1}{2}(H + B)$, and $C'' = \frac{1}{2}(H + C)$. Let $\mathcal{S}$ a circle with the line segment $\overline{A'A''}$ as diameter. Show that $B', B'', C', C'' \in \mathcal{S}$ using Exercise 3.7 and Thales' Theorem (Theorem 3.6).

Solution: By Exercise 3.7, we see that $A'B'A''B''$ and $A'C'A''C''$ are rectangles. Thus, $\ell_{A''B'}$ is perpendicular to $\ell_{A'B'}$. By Thales theorem, we obtain that $B' \in \mathcal{S}$. Similarly, we have $\ell_{A''B''} \perp \ell_{A'B''}$, $\ell_{A''C'} \perp \ell_{A'C'}$, and $\ell_{A''C''} \perp \ell_{A'C''}$. Thus, we conclude that B'', C', C'' belong to the circle $\mathcal{S}$.

2. (Continued) Show that line segments $\overline{B'B''}$ and $\overline{C'C''}$ are also diameters of $\mathcal{S}$. Let D, E, F be the feet of the altitudes of ℓ_A, ℓ_B, ℓ_C. Using Thales' Theorem, deduce that $D, E, F \in \mathcal{S}$.

Solution: Since $|A'A''| = |B'B''| = |C'C''|$ by Exercise 3.7, $\overline{A'A''}$ is a diameter of $\mathcal{S}$, and
$$\frac{1}{2}(A' + A'') = \frac{1}{4}(A + B + C + H) = \frac{1}{2}(B' + C'') = \frac{1}{2}(C' + C''),$$
we see that $\overline{B'B''}$ and $\overline{C'C''}$ are also diameters of $\mathcal{S}$. Since A'' is on the altitude $\ell_{AH} = \ell_A$, we have $\ell_{A''D} \perp \ell_{DA'} = \ell_{BC}$. By Thales theorem, D belongs to the circle $\mathcal{S}$. Similarly,
$$\ell_{B''E} = \ell_B \perp \ell_{B'E} = \ell_{AC}, \quad \ell_{C''F} = \ell_C \perp \ell_{C'F} = \ell_{AB},$$
which implies $E, F \in \mathcal{S}$.

3. Let $A, B \in \mathbb{R}^2$ with $A \neq B$. Find the minimum of $d(P, A) + d(P, B)$ for $P \in \mathbb{R}^2$.

Solution: By triangle inequality, for any P in $\mathbb{R}^2$, we have
$$d(P, A) + d(P, B) \geqslant d(A, B).$$
Furthermore, if $P = A$, then $d(P, A) + d(P, A) = d(A, B)$. Thus, the minimum is $d(A, B)$.

4. Exercise 3.12

Solution: Let R be the point in $\ell_{AC} \cap \ell_{BD}$. For any P, it follows from triangle inequality that
$$d(R, A) + d(R, C) = d(A, C) \leqslant d(P, A) + d(P, C),$$
$$d(R, B) + d(R, D) = d(B, D) \leqslant d(P, B) + d(P, D).$$
Therefore, the minimum is $d(A, C) + d(B, D)$ and it is attained when $P = R$.

5. Let $X, Y, A \in \mathbb{R}^2 \setminus \{O\}$. Suppose $X \cdot A = 0$ and $Y \cdot A = 0$. Show that O, X, Y are collinear.

Solution: Suppose O, X, Y are not collinear, then there exist $a, b \in \mathbb{R}$ such that $A = aX + bY$. Then,
$$A \cdot A = (aX + bY) \cdot A = a(X \cdot A) + b(Y \cdot A) = 0.$$
Thus, $A = O$, which is a contradiction. Therefore, O, X, Y are collinear.