MATH 403 FALL 2021: EXAM 2 PRACTICE PROBLEMS

1. Definitions

(a) One-to-one, onto, bijection maps.

(b) Fixed points of a map.

(c) Two maps are equal.

(d) Central dilatation, translation, dilatation, central reflection.

(e) Conjugate of α by μ.

(f) Groups, subgroups, Abelian groups, homomorphism, isomorphism.

(g) Scalar product of X, Y, the length of X, the distance between X, Y.

2. Examples

(a) Find one-to-one but not onto maps.

(b) Find onto but not one-to-one maps.

(c) Find two maps α, β such that $\alpha \circ \beta \neq \beta \circ \alpha$

(d) Find two maps α, β such that $\alpha \circ \beta = \beta \circ \alpha$

(e) Find a set of transformations that does not form a group.

(f) Find a bijection $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ that does not map a line to a line.

3. Proof or Disproof

3.1. Translations and Central dilatations – Basic.

(a) $\tau_A \tau_B = \tau_{A+B}$, $(\tau_A)^{-1} = \tau_{-A}$, $\delta_{C,r}\delta_{C,s} = \delta_{C,rs}$, $(\delta_{C,r})^{-1} = \delta_{C,1/r}$.

(b) $\delta_r \circ \tau_A = \tau_{rA} \circ \delta_r$

(c) $\delta_{A+C,r} = \tau_C \delta_{A,r} \tau_C^{-1}$.

(d) $\sigma_C^2 = \mathrm{Id}$

(e) $\sigma_C \sigma_D$ is a translation.

(f) $\delta_{C,r} \circ \delta_{C,r}(X) = X$ for all X implies $r = 1$ or $r = -1$.

(g) If $\delta_{C,r}(X) = Y$ then C, X, Y are collinear.

(h) A translation and a central dilatation commute.

(i) For a dilatation α, $\alpha(\ell_{AB}) = \ell_{\alpha(A)\alpha(B)}$.

(j) For a dilatation α, $\ell_{AB} \mathbin{/\!/} \ell_{\alpha(A)\alpha(B)}$.

3.2. Translations and Central dilatations – Applications.

(a) Translation preserves the centroid of three points.

(b) Every dilatation has at least one fixed point.

(c) Central dilatations preserve midpoints.

(d) Let $\alpha = \delta_{A,2}$. If A, B, C form a triangle and G is the centroid, then $B, G, C, \alpha(G)$ form a parallelogram.

(e) $\sigma_D \sigma_C \sigma_B \sigma_A = \mathrm{Id}$ if and only if $ABCD$ is a parallelogram.

3.3. **Group theory.**

(a) Let V be a set and G be the set of all bijections $\alpha : V \to V$. Show that G with composition is a group.

(b) Let G be a group and H be a subset. Then, H is a group if and only if $ab^{-1} \in H$ for all $a, b \in H$.

(c) The set of all translations forms a group.

(d) For fixed $A \neq O$, $\{\tau_{cA} : c \in \mathbb{R}\}$ is a group.

(e) For fixed $C \in \mathbb{R}^2$, $\{\delta_{C,r} : r \in \mathbb{R}, r > 0\}$ is a group.

(f) The set of all central dilatations forms a group.