MATH 403 FALL 2021: HOMEWORK 4

INSTRUCTOR: DAESUNG KIM
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1. Let $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}^2$ be two maps.
   (a) Show that if α and β are one-to-one, then so is $\alpha \circ \beta$.
   (b) Show that if α and β are onto, then so is $\alpha \circ \beta$.
   (c) Conclude that if α and β are bijections, then so is $\alpha \circ \beta$.
2. Let $\alpha, \beta, \gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be maps. Show that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.
3. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection. Show that the inverse of α is unique. That is, suppose there are two maps $\beta, \gamma : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\alpha \circ \beta = \beta \circ \alpha = \alpha \circ \gamma = \gamma \circ \alpha = \mathrm{Id}$. Then, show that $\beta = \gamma$.
4. (Will not be graded) Let $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}^2$ be two maps.
   (a) Give an example that α is not one-to-one and β is not onto but $\alpha \circ \beta$ is a bijection.
   (b) Give an example that $\alpha \circ \beta = \mathrm{Id}$ but α is not the inverse of β.
   (Hint: find such examples for $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of all natural numbers.)
5. Let $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection. For a map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, the conjugate of α by μ is denoted by $\overline{\alpha}$ and defined by $\overline{\alpha} = \mu \circ \alpha \circ \mu^{-1}$.
   (a) Show that $\overline{\alpha \circ \beta} = \overline{\alpha} \circ \overline{\beta}$.
   (b) Show that if α is a bijection, then so is $\overline{\alpha}$.
   (c) Let α be a bijection. Show that $(\overline{\alpha})^{-1} = \overline{\alpha^{-1}}$.
6. Let $A, C \in \mathbb{R}^2$ and $r \in \mathbb{R} \setminus \{0\}$.
   (a) The map $\delta_{A+C,r}$ is the conjugate of $\delta_{A,r}$ by τ_C. That is, $\delta_{A+C,r} = \tau_C \circ \delta_{A,r} \circ (\tau_C)^{-1}$.
   (b) $\delta_{C,r}$ maps a line to a parallel line. That is, suppose ℓ is a line. Show that $\delta_{C,r}(\ell)$ is a line and parallel to ℓ.