MATH 403 FALL 2021: HOMEWORK 4 SOLUTION

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1. Let $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}^2$ be two maps.
   (a) Show that if α and β are one-to-one, then so is $\alpha \circ \beta$.
   (b) Show that if α and β are onto, then so is $\alpha \circ \beta$.
   (c) Conclude that if α and β are bijections, then so is $\alpha \circ \beta$.

Solution:
(a) If $\alpha\beta(X) = \alpha\beta(Y)$, then $\beta(X) = \beta(Y)$ since α is one-to-one. Since β is one-to-one, $X = Y$.
(b) Let $Y \in \mathbb{R}^2$, then there exists $Z \in \mathbb{R}^2$ such that $Y = \alpha(Z)$ since α is onto. Since β is onto, there exists X such that $Z = \beta(X)$. Thus, $Y = \alpha\beta(X)$, which implies that $\alpha\beta$ is onto.
(c) This follows from (a) and (b).

2. Let $\alpha, \beta, \gamma : \mathbb{R}^2 \to \mathbb{R}^2$ be maps. Show that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.

Solution: Let $X \in \mathbb{R}^2$. Then,
$$(\alpha\beta)\gamma(X) = (\alpha\beta)(\gamma(X)) = \alpha(\beta(\gamma(X))) = \alpha(\beta\gamma(X)) = \alpha \circ (\beta \circ \gamma)(X).$$

3. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection. Show that the inverse of α is unique. That is, suppose there are two maps $\beta, \gamma : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\alpha \circ \beta = \beta \circ \alpha = \alpha \circ \gamma = \gamma \circ \alpha = \mathrm{Id}$. Then, show that $\beta = \gamma$.

Solution: By Problem 2 and the definition of inverse map, we have
$$\beta = \beta \circ \mathrm{Id} = \beta \circ (\alpha \circ \gamma) = (\beta \circ \alpha) \circ \gamma = \mathrm{Id} \circ \gamma = \gamma.$$

4. (Will not be graded) Let $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}^2$ be two maps.
   (a) Give an example that α is not one-to-one and β is not onto but $\alpha \circ \beta$ is a bijection.
   (b) Give an example that $\alpha \circ \beta = \mathrm{Id}$ but α is not the inverse of β.
   (Hint: find such examples for $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of all natural numbers.)

Solution: Define α, β by
$$\alpha(X) = \begin{cases} (n-1, 0), & X = (n, 0), n \in \mathbb{N}, \\ X, & \text{otherwise}, \end{cases} \qquad \beta(X) = \begin{cases} (n+1, 0), & X = (n, 0), n \in \mathbb{N} \cup \{0\}, \\ X, & \text{otherwise}. \end{cases}$$
Note that α is not one-to-one because $\alpha((1,0)) = \alpha(O) = O$, and β is not onto because there is not X such that $O = \beta(X)$. If $X = (n, 0)$ with $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\beta(X)) = \alpha((n+1, 0)) = (n, 0) = X$. Thus, $\alpha\beta = \mathrm{Id}$.

5. Let $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection. For a map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, the conjugate of α by μ is denoted by $\overline{\alpha}$ and defined by $\overline{\alpha} = \mu \circ \alpha \circ \mu^{-1}$.
   (a) Show that $\overline{\alpha \circ \beta} = \overline{\alpha} \circ \overline{\beta}$.
   (b) Show that if α is a bijection, then so is $\overline{\alpha}$.
   (c) Let α be a bijection. Show that $(\overline{\alpha})^{-1} = \overline{\alpha^{-1}}$.

Solution:
(a) $\overline{\alpha \circ \beta} = \mu \circ (\alpha \circ \beta)\mu^{-1} = (\mu \circ \alpha \circ \mu^{-1}) \circ (\mu \circ \beta \circ \mu^{-1}) = \overline{\alpha} \circ \overline{\beta}$.

(b) This follows from Problem 1 (c).

(c) Note that $\overline{\mathrm{Id}} = \mathrm{Id}$. By Part (a),

$$\overline{\alpha} \circ \overline{\alpha^{-1}} = \overline{\alpha \circ \alpha^{-1}} = \overline{\mathrm{Id}} = \mathrm{Id},$$

and

$$\overline{\alpha^{-1}} \circ \overline{\alpha} = \overline{\alpha^{-1} \circ \alpha} = \overline{\mathrm{Id}} = \mathrm{Id}.$$

6. Let $A, C \in \mathbb{R}^2$ and $r \in \mathbb{R} \setminus \{0\}$.
   (a) The map $\delta_{A+C,r}$ is the conjugate of $\delta_{A,r}$ by τ_C. That is, $\delta_{A+C,r} = \tau_C \circ \delta_{A,r} \circ (\tau_C)^{-1}$.
   (b) $\delta_{C,r}$ maps a line to a parallel line. That is, suppose ℓ is a line. Show that $\delta_{C,r}(\ell)$ is a line and parallel to ℓ.

Solution:

(a) For $X \in \mathbb{R}^2$,

$$\begin{aligned}\tau_C \circ \delta_{A,r}(X) &= \tau_C((1-r)A + rX) \\ &= (1-r)(A+C) + r(X+C) \\ &= \delta_{A+C,r}(X+C) \\ &= \delta_{A+C,r} \circ \tau_C(X).\end{aligned}$$

(b) Suppose $\ell = \ell_{AB}$ with distinct A, B. If $P \in \ell$, then $P = (1-s)A + sB$ for some s. Then,

$$\begin{aligned}\delta_{C,r}(P) &= (1-r)C + r((1-s)A + sB) \\ &= (1-s)((1-r)C + rA) + s((1-r)C + rB) \\ &= (1-s)\delta_{C,r}(A) + s\delta_{C,r}(B).\end{aligned}$$

Since $\delta_{C,r}$ is one-to-one, $A' = \delta_{C,r}(A)$ and $B' = \delta_{C,r}(B)$ are distinct. Thus, $\delta_{C,r}(P)$ lies on the line $\ell_{A'B'}$. Since

$$A' - B' = r(A - B),$$

the lines are parallel.