MATH 403 FALL 2021: HOMEWORK 1 SOLUTION

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1. For any vectors $A = (a_1, a_2)$, $B = (b_1, b_2)$ and a real number $r \in \mathbb{R}$, we have seen in class that
$$A + B = (a_1 + b_1, a_2 + b_2), \quad rA = (ra_1, ra_2).$$
Using these, show the eight properties of addition and scalar multiplications labeled in the textbook by (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4).

Solution: Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $r, s \in \mathbb{R}$.

(A1): By definition, $A + B = (a_1 + b_1, a_2 + b_2)$ and $B + A = (b_1 + a_1, b_2 + a_2)$. Since real numbers are commutative (that is, $a + b = b + a$), we have $a_1 + b_1 = b_1 + a_1$ and $a_2 + b_2 = b_2 + a_2$. Thus, $A + B = B + A$.

(A2): Since $a_1 + (b_1 + c_1) = (a_1 + b_1) + c_1$ and $a_2 + (b_2 + c_2) = (a_2 + b_2) + c_2$, we have $A + (B + C) = (A + B) + C$.

(A3): Since $0 + a_1 = a_1 = a_1 + 0$ and $0 + a_2 = a_2 = a_2 + 0$, we have $O + A = A = A + O$.

(A4): Note that $-A = (-1)A = (-a_1, -a_2)$. Since $a_1 + (-a_1) = 0$ and $a_2 + (-a_2) = 0$, we have $A + (-A) = O$.

(M1): Since $(r + s)a_1 = ra_1 + sa_1$ and $(r + s)a_2 = ra_2 + sa_2$, we have $(r + s)A = rA + sA$.

(M2): Since $r(a_1 + b_1) = ra_1 + rb_1$ and $r(a_2 + b_2) = ra_2 + rb_2$, we have $r(A + B) = rA + rB$.

(M3): Since $r(sa_1) = (rs)a_1$ and $r(sa_2) = (rs)a_2$, we have $r(sA) = (rs)A$.

(M4): Since $1 \cdot a_1 = a_1$ and $1 \cdot a_2 = a_2$, we have $1A = A$.

2. Using the eight properties only, Do Exercise 1.2, 1.3, 1.4, 1.5.

Solution: (Exercise 1.2): Suppose $A + B = A + C$. By (A2), we have
$$(-A) + (A + B) = ((-A) + A) + B = ((-A) + A) + C = (-A) + (A + C).$$
Using (A4) and (A3),
$$((-A) + A) + B = O + B = B, \quad ((-A) + A) + C = O + C = C.$$
Therefore, we conclude $B = C$.

(Exercise 1.3): By (A3), $O + O = O$. By (M2), we have
$$rO = r(O + O) = rO + rO.$$
By (A3) again, $rO = rO + O$. By Exercise 1.2 (which follows only from (A1-4) and (M1-4)), we conclude
$$rO + O = rO + rO \text{ implies } rO = O.$$

(Exercise 1.4): By (M1), we have
$$0A = (0 + 0)A = 0A + 0A.$$
By (A3), $0A = 0A + O$. By Exercise 1.2, we conclude
$$0A + O = 0A + 0A \text{ implies } 0A = O.$$

(Exercise 1.5): Assume that $rA = O$ and $r \neq 0$. By (M3) and (M4),

$$\begin{aligned} O &= \frac{1}{r}O \qquad \text{by Exercise 1.3} \\ &= \frac{1}{r}(rA) = 1A \qquad \text{by (M3)} \\ &= A \qquad \text{by (M4).} \end{aligned}$$

3. Exercise 1.6

Solution: Let M be the midpoint of A and B. Then, by definition, we have

$$M - A = \overrightarrow{AM} = \overrightarrow{MB} = B - M.$$

By (A2),

$$\begin{aligned} (M + M) + (-A) &= M + (M - A) \qquad \text{by (A2)} \\ &= M + (B - M) \qquad \text{by definition of midpoint} \\ &= (B - M) + M \qquad \text{by (A1)} \\ &= B + ((-M) + M) \qquad \text{by (A2)} \\ &= B + O = B \qquad \text{by (A4) and (A3).} \end{aligned}$$

Similarly, we obtain $2M = A + B$. Thus, $M = \frac{1}{2}(2M) = \frac{1}{2}(A + B)$ as desired. If $A = B$, then $M = \frac{1}{2}(A + B) = A = B$. So, the midpoint is well-defined when $A = B$, that is, A and B do not have to be distict. In this case, $M = A = B$.

4. Exercise 1.7

Solution: (a): It suffices to show that $M_1 + M_3 = M_2 + M_4$ by definition. Indeed,

$$\begin{aligned} M_1 + M_3 &= \frac{1}{2}(A + B) + \frac{1}{2}(C + D) = \frac{1}{2}(A + B + C + D), \\ M_2 + M_4 &= \frac{1}{2}(B + C) + \frac{1}{2}(D + A) = \frac{1}{2}(A + B + C + D). \end{aligned}$$

Thus, $M_1M_2M_3M_4$ defines a parallelogram.

(b): Suppose $A = B$. Then, $M_1 = A = B$. In this case, $ABCD$ forms a triangle. Still, $M_1M_2M_3M_4$ defines a parallelogram.

Suppose $A = B = C$, then $M_1 = M_2 = A = B = C$ and $M_3 = M_4$. In this case, $M_1M_2M_3M_4$ forms a line segment, which is also a parallelogram in our sense.

5. Exercise 1.8

Solution: We have

$$B' - A' = \frac{1}{2}(A + C - B - C) = \frac{1}{2}(A - B).$$

By definition, $\ell_{AB} = \ell_{A'B'}$. Similarly, we have $\ell_{BC} = \ell_{B'C'}$ and $\ell_{CA} = \ell_{C'A'}$. Note that

$$C' - A = \frac{1}{2}(A + B) - A = \frac{1}{2}(B - A) = A' - B.$$

Suppose A', B', C' are given and they are not on a line. Then $A'B'C'$ forms a triangle. Suppose ABC defines a triangle such that A' is the midpoint of B, C, B' is the midpoint of A, C and C' is the midpoint of A, B. By the previous statement, the line ℓ_{AB} passes through C' and is parallel to $\ell_{A'B'}$. Similarly, ℓ_{BC} passes through A' and is parallel to $\ell_{B'C'}$. Since ℓ_{AB} and ℓ_{BC} meet at B, the point B is determined by A', B', C'. Similarly, A and C are determined too. In other words, for given A', B', C', there is only one $\triangle ABC$.

In particular, A, B, C are uniquely determined by A', B', C'. Since $A + B = 2C'$, $B + C = 2A'$, and $C + A = 2B'$, we have $A + B + C = A' + B' + C'$. Thus, A, B, C can be written as

$$A = (A + B + C) - (B + C) = (A' + B' + C') - 2A',$$
$$B = (A + B + C) - (C + A) = (A' + B' + C') - 2B',$$
$$C = (A + B + C) - (A + B) = (A' + B' + C') - 2C'.$$

6. Let $a, b, c, d, p, q \in \mathbb{R}$. Consider a system of linear equations

$$\begin{cases} ax + by = p, \\ cx + dy = q. \end{cases}$$

(Here, x and y are unknown variables.) Show that there exists a unique solution (x, y) for the system if and only if $ad - bc \neq 0$.

Solution: Suppose $ad \neq bc$. By multiplying d on the first equation and b on the second,we get $adx + bdy = pd$ and $bcx + bdy = qb$. By subtracting, we have $(ad - bc)x = pd - qb$. Since $ad - bc \neq 0$, we obtain $x = (pd - qb)/(ad - bc)$. Since either $b \neq 0$ or $d \neq 0$, y is also determined by x. Therefore, there exists a unique solution.

Suppose $ad = bc$. If $a = 0$, then either $b = 0$ or $c = 0$. Case 1: $a = b = 0$. If $p = 0$, then there are infinitely many solutions. If $p \neq 0$, then there is no solution. Case 2: $a = c = 0$. Then we have two equations $by = p$ and $dy = q$. Since x can be any number for this system, there are no solutions or infinitely many solution.

Now assume that $ad = bc$ and none of a, b, c, d are zero. By multiplying d on the first equation and b on the second, we get $adx + bdy = pd$ and $bcx + bdy = qb$. If $pd = qb$, then there are infinitely many solutions. Otherwise, there is no solution.

Therefore, if there exists a unique solution, then $ad \neq bc$.

7. Exercise 1.11

Solution: Consider $\triangle ABC$. Then the centroid G of ABC is $G = \frac{1}{3}(A + B + C)$. Let A', B', C' be the midpoints of B and C, C and A, and A and B respectively. Then, the centroid G' of $A'B'C'$ is

$$G' = \frac{1}{3}(A' + B' + C') = \frac{1}{3}\left(\frac{1}{2}(B + C) + \frac{1}{2}(C + A)\frac{1}{2}(A + B)\right) = \frac{1}{3}(A + B + C).$$

8. Exercise 1.13

Solution: Let M be the midpoint of B, C and N the midpoint of C, D. Let P be the intersection of ℓ_{AM} and ℓ_{BD}, and Q the intersection of ℓ_{AN} and ℓ_{BD}. We want to show that P and Q trisect the diagonal $\overline{BD}$.

Since P lies on ℓ_{AM} and ℓ_{BD}, there exist $s, t \in \mathbb{R}$ such that

$$P = (1 - t)B + tD = (1 - s)A + sM = (1 - s)A + \frac{s}{2}B + \frac{s}{2}C.$$

Since $ABCD$ is a parallelogram, we have $A + C = B + D$. Replacing A with $B + D - C$, we obtain

$$(\frac{s}{2} - t)B + (s + t - 1)D = (\frac{3s}{2} - 1)C.$$

If $\frac{3s}{2} - 1 \neq 0$, then we see that C is on the line ℓ_{BD}. In this case, A and C are on the diagonal through B, D so it is trivial. If C is not on the line ℓ_{BD}, then $\frac{3s}{2} = 1$, that is $s = 2/3$. Thus, $t = 2/3$ and so P trisects $\overline{BD}$. Similarly, Q trisects $\overline{BD}$.