MATH 403 LECTURE NOTE
WEEK 2

DAESUNG KIM

1. Parallelograms and Centroid (Sec 1.6-8)

Recall that four points A, B, C, D define a parallelogram $ABCD$ if $A + C = B + D$. We do not assume that A, B, C, D are distinct. Thus, our definition generalizes parallelograms. For instance, if $A = B$, then $C = D$ by the equation. Thus, the parallelogram $ABCD$ is just a line segment $\overline{AD} = \overline{BC}$.

Proposition 1.1. *A quadrilateral is a parallelogram if and only if the diagonals bisect each other.*

Proof. Suppose a quadrilateral $ABCD$ is a parallelogram. From the defining equation, we have $\frac{1}{2}(A+C) = \frac{1}{2}(B+D)$. Since $A+C$ and $B+D$ are the diagonals of $ABCD$, we see that $\frac{1}{2}(A+C)$ and $\frac{1}{2}(B+D)$ are the midpoints of them. Thus, the diagonals bisect each other.

Suppose the diagonals of a quadrilateral $ABCD$ bisect each other. Then, vectors A, B, C, D should satisfy $\frac{1}{2}(A+C) = \frac{1}{2}(B+D)$, which implies the defining equation. Thus, $ABCD$ is a parallelogram. ∎

Definition 1.2. *Consider a triagle $\triangle ABC$. A median is a line joining a vertex to the midpoint of the opposite side.*

Theorem 1.3. *The medians of a triagle are concurrent. That is, the medians intersect in one point.*

Proof. By translation, we assume that $A = O$. We also assume that B and C are not on the same line. Otherwise, ABC does not form a triangle. Let A' be the midpoint of B and C, and B' be the midpoint of C and A. Consider $P \in \ell_{AA'}$ and $Q \in \ell_{BB'}$, then

$$P = P(t) = (1-t)A + tA' = (1-t)A + \frac{t}{2}B + \frac{t}{2}C = \frac{t}{2}B + \frac{t}{2}C,$$
$$Q = Q(s) = (1-s)B + sB' = (1-s)B + \frac{s}{2}C + \frac{s}{2}A = (1-s)B + \frac{s}{2}C.$$

Thus, if $P = Q$, then

$$(1 - s - \frac{t}{2})B = \frac{t-s}{2}C.$$

Since B and C are not on the same line, the coefficients should be zero. Thus, $t = s = \frac{2}{3}$. Thus, the intersection of the two medians is

$$\ell_{AA'} \cap \ell_{BB'} = \left\{ \frac{1}{3}(A+B+C) \right\}.$$

Since the point is on the other median $\ell_{CC'}$, where $C' = \frac{1}{2}(A+B)$ (why?), the proof is complete. ∎

Definition 1.4. *The* centroid *of a triangle is a unique point in the intersection of three medians. According to the proof of the previous theorem, the centroid can be written as*

$$G = \frac{1}{3}(A+B+C).$$

In general,

Definition 1.5. *The centroid G of $\{A_1, A_2, \cdots, A_n\} \subset \mathbb{R}^2$ is defined by*

$$G = \frac{1}{n}(A_1 + A_2 + \cdots + A_n).$$

Note that G is the midpoint if $n = 2$ and the centroid of a triangle $\triangle ABC$ if $n = 3$.

Theorem 1.6. *Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$. Let G_1 be the centroid of A_2, A_3, A_4, G_2 be the centroid of A_1, A_3, A_4, and so on. Let G be the centroid of A_1, A_2, A_3, A_4. Then,*

$$G \in \ell_{A_1G_1} \cap \ell_{A_2G_2} \cap \ell_{A_3G_3} \cap \ell_{A_4G_4}.$$

Proof. By symmetry, it suffices to show that $G \in \ell_{A_1G_1}$. Every point P on the line $\ell_{A_1G_1}$ can be written as

$$P = (1-t)A_1 + tG_1 = (1-t)A_1 + \frac{t}{3}A_2 + \frac{t}{3}A_3 + \frac{t}{3}A_4$$

for some $t \in \mathbb{R}$. If $t = \frac{1}{4}$, then one can see that $G = P$. Thus, $G \in \ell_{A_1G_1}$. ∎

Remark 1.7. *Do the lines $\ell_{A_iG_i}$, $i = 1, 2, 3, 4$ intersect in a single point? In other words,*

$$\ell_{A_1G_1} \cap \ell_{A_2G_2} \cap \ell_{A_3G_3} \cap \ell_{A_4G_4} = \{G\}?$$

Proposition 1.8. *Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$. Let M_1 be the midpoint of A_1, A_2 and M_2 be the midpoint of A_3, A_4. Let G be the centroid of A_1, A_2, A_3, A_4. Then, $G \in \ell_{M_1M_2}$.*

Proof. Every point P on $\ell_{M_1M_2}$ can be written as

$$P = (1-t)M_1 + tM_2 = \frac{1-t}{2}A_1 + \frac{1-t}{2}A_2 + \frac{t}{2}A_3 + \frac{t}{2}A_4$$

for some $t \in \mathbb{R}$. If $t = \frac{1}{2}$, then $P = G$. ∎

2. Centroid of mass-points (Sec 1.10)

Proposition 2.1. *Let $p, q \in \mathbb{N}$. Let $A_1, A_2, \cdots, A_p \in \mathbb{R}^2$ and $B_1, B_2, \cdots, B_q \in \mathbb{R}^2$. Let U be the centroid of $A_1, \cdots, A_p$ and V be the centroid of $B_1, \cdots, B_q$. Let G be the centroid of $A_1, A_2, \cdots, A_p, B_1, \cdots, B_q$. Then,*

$$G = \frac{p}{p+q}U + \frac{q}{p+q}V.$$

In particular, $G \in \ell_{UV}$.

Suppose A_i and B_j for $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ have the same weight, say 1. Then, we think that all the weights of A_i are concentrated at U, and all the weights of B_j at V. Then the centroid of G can be understood as the weighted midpoint of U and V.

Definition 2.2. *Let $a \in \mathbb{R}$ and $A \in \mathbb{R}^2$. We call a pair (a, A) a weighted point or a mass-point.*

Definition 2.3. *The centroid G of mass-points $(a_1, A_1), \cdots, (a_n, A_n)$ is defined by*

$$G = \frac{1}{a_1 + \cdots + a_n}(a_1A_1 + \cdots + a_nA_n).$$

Definition 2.4. *Let (a, A) and (b, B) be mass-points with $a + b \neq 0$. We define*

$$(a, A) + (b, B) := (a + b, \frac{a}{a+b}A + \frac{b}{a+b}B).$$

Note that one can think G as a mass-point with weight $(a_1 + \cdots + a_n)$ and

$$((a_1 + \cdots + a_n), G) = (a_1, A_1) + \cdots + (a_n, A_n).$$

Let $(a, A), (b, B)$ be mass-points with $a + b \neq 0$. Then,

$$P = \frac{a}{a+b}A + \frac{b}{a+b}B$$

lies on the line ℓ_{AB}. In particular, we have

$$P - A = \frac{b}{a+b}(B - A), \quad P - B = \frac{a}{a+b}(A - B).$$

Thus, we have

$$P - A = -\frac{b}{a}(P - B), \quad \text{if } a \neq 0,$$
$$P - B = -\frac{a}{b}(P - A), \quad \text{if } b \neq 0.$$

Definition 2.5. *We introduce the symbol* $\frac{P-A}{P-B} = t$ *to denote* $P - A = t(P - B)$*. This does not imply that we can divide a vector by another vector. It is just for notational convenience.*

Theorem 2.6. *Let* $(a, A), (b, B), (c, C)$ *be mass-points with* $a + b + c \neq 0$, $a + b \neq 0$, $b + c \neq 0$, *and* $c + a \neq 0$. *Let*

$$A' = \frac{bB + cC}{b + c}, \quad B' = \frac{cC + aA}{c + a}, \quad C' = \frac{aA + bB}{a + b}.$$

Then, the lines $\ell_{AA'}$, $\ell_{BB'}$ *and* $\ell_{CC'}$ *are concurrent and*

$$\ell_{AA'} \cap \ell_{BB'} \cap \ell_{CC'} = \{G\}$$

where G *is the centroid of* $(a, A), (b, B), (c, C)$.

Proof. It suffices to show that $G \in \ell_{AA'}$ by symmetry. In other words, we need to show that there exists $t \in \mathbb{R}$ such that

$$G = (1 - t)A + tA'$$

Indeed, since

$$G = \frac{a}{a + b + c}A + \frac{b}{a + b + c}B + \frac{c}{a + b + c}C = (1 - t)A + \frac{tb}{b + c}B + \frac{tc}{b + c}C = (1 - t)A + tA',$$

it is natural to take $t = 1 - a/(a + b + c)$ by comparing the coefficients of A. Then, one can see that

$$\frac{tb}{b + c} = \frac{b}{a + b + c}, \quad \frac{tc}{b + c} = \frac{c}{a + b + c},$$

as desired. ∎

3. Baricentric coordinates (Sec 1.11)

Theorem 3.1. *Let* $A, B, C \in \mathbb{R}^2$ *be points which do not lie on a line. For every* P *in the plane, there is a unique representation*

$$P = aA + bB + cC$$

with $a + b + c = 1$.

Proof. Subtracting A on the both sides, we have

$$P - A = (aA + bB + cC) - (aA + bA + cA) = b(B - A) + c(C - A).$$

Here, I used the assumption that $a + b + c = 1$.

Recall that if X and Y are vectors such that $\ell_{OX} \neq \ell_{OY}$, then for every $Z \in \mathbb{R}^2$, there is a unique representation $Z = rX + sY$.

Here, we assume that A, B, C are not on the same line. In other words, if we let $X = B - A$ and $Y = C - A$, then $\ell_{OX} \neq \ell_{OY}$. Thus, there is a unique representation for $Z := P - A$ in terms of X and Y. That is,

$$P - A = b(B - A) + c(C - A)$$

for some $b, c \in \mathbb{R}$. Thus, we have a unique representation

$$P = (1 - b - c)A + bB + cC = aA + bB + cC.$$

∎

Definition 3.2. *Let* $A, B, C \in \mathbb{R}^2$ *be points which do not lie on a line. The barycentric coordinate of* P *with respect to* A, B, C *is defined by* (a, b, c) *such that* $a, b, c \in \mathbb{R}$ *with* $a + b + c = 1$ *and*

$$P = aA + bB + cC.$$

Remark 1. *Let* $A, B, C \in \mathbb{R}^2$ *be points which do not lie on a line. Every point* P *can be also written as*

$$P = \frac{a}{a + b + c}A + \frac{b}{a + b + c}B + \frac{c}{a + b + c}C.$$

Such a, b, c *are also unique. Then,* P *is the centroid of mass-points* (a, A), (b, B) *and* (c, C).

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu