MATH 403 LECTURE NOTE
WEEK 11

DAESUNG KIM

1. PROJECTIONS (SEC. 3.5)

Let $X, Y \in \mathbb{R}^2$, $X \neq O$. The (orthogonal) projection of Y to X is a scalar multiple of X (that is, a vector lying on the line ℓ_{OX}) such that $Y - cX$ is orthogonal to ℓ_{OX}. Thus, we have

$$(Y - cX) \cdot X = 0,$$

which implies $c = X \cdot Y / |X|^2$. We define

$$\mathrm{Proj}_X Y = \frac{X \cdot Y}{|X|^2} X.$$

We call a vector X is a unit vector if $|X| = 1$. If X is a unit vector, then $\mathrm{Proj}_X Y = (X \cdot Y)X$.

2. ANGLES (SEC. 3.6)

Let $X, Y \in \mathbb{R}^2$ and θ be the angle between X and Y. This angle is chosen so that $-\pi < \theta \leqslant \pi$. We choose the sign of the angle according to the counterclockwise orientation from X to Y. Then, the angle can be written as

$$\cos\theta = \frac{X \cdot Y}{|X||Y|}.$$

We denote by $\theta = \measuredangle(X, Y)$.

For a triangle $\triangle ABC$, the area is denoted by $|\triangle ABC|$.

Proposition 2.1. *Let $X, Y \in \mathbb{R}^2$ be nonzero, then*

$$|\triangle OXY| = \frac{1}{2}|X||Y||\sin\measuredangle(X, Y)|.$$

Proof. Note that $\mathrm{Proj}_X Y = \cos\theta |Y| X/|X|$, where $\theta = \measuredangle(X, Y) \in (-\pi, pi]$. Then,

$$|\triangle OXY| = \frac{1}{2}|X||Y - \mathrm{Proj}_X Y| = \frac{1}{2}|X||Y| \left| \frac{Y}{|Y|} - \cos\theta \frac{X}{|X|} \right|.$$

Let $Z = X/|X|$ and $W = Y/|Y|$, then $|Z| = |W| = 1$ and $Z \cdot W = \cos\theta$. Thus,

$$|W - \cos\theta Z|^2 = |W|^2 - 2\cos\theta Z \cdot W + \cos^2\theta |Z|^2 = 1 - \cos^2\theta = \sin^2\theta,$$

which finishes the proof. ∎

For $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, we define the determinant of X, Y by

$$\det(X, Y) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1.$$

Proposition 2.2. *Let $X, Y \in \mathbb{R}^2$ and $\theta = \measuredangle(X, Y)$, then*

$$\det(X, Y) = |X||Y| \sin\theta.$$

Proof. Let Z be the vector obtained from X by rotating $\pi/2$ counterclockwise. If $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, then $Z = (-x_2, x_1)$ and $Y \cdot Z = |X||Y|\cos(\pi/2 - \theta) = |X||Y|\sin\theta = \det(X, Y)$. ∎

Proposition 2.3. *Let $X, Y \in \mathbb{R}^2$.*

(1) $\det(X, Y) > 0$ *if and only if* $\measuredangle(X, Y) \in (0, \pi)$.

(2) $\det(X, Y) < 0$ *if and only if* $\measuredangle(X, Y) \in (-\pi, 0)$.

(3) $\det(X, Y) = 0$ *if and only if* $\measuredangle(X, Y) \in \{0, \pi\}$.
(4) $|\det(X, Y)| = 2|\triangle OXY|$.

Proposition 2.4. *Let* $X, Y \in \mathbb{R}^2$*, then*

$$|X - Y|^2 = |X|^2 + |Y|^2 - 2\cos\measuredangle(X, Y)|X||Y|.$$

Theorem 2.5 (Heron's formula)**.** *Consider a triangle* $\triangle OXY$*. Let* $a = |X|$*,* $b = |Y|$*,* $c = |X - Y|$*, and* $s = \frac{1}{2}(a + b + c)$*. Then,*

$$|\triangle OXY| = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. HW. ■

3. Equation of a line (Sec. 3.7)

Consider a line ℓ through P and orthogonal to a given vector $N \neq O$. Then, $X \in \ell$ if and only if $(X - P) \cdot N = 0$. The equation $X \cdot N = P \cdot N$ is called the equation of the line ℓ.

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu