MATH 403 FALL 2021: HOMEWORK 3 SOLUTION

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1. Exercise 1.17

Solution: Let $A, B, C \in \mathbb{R}^2$ and $a \in \mathbb{R}$ be given. Consider

$$\ell = \{P \in \mathbb{R}^2 : P = aA + bB + cC, a + b + c = 1\}.$$

Since $P = b(B+a(A-C))+(1-b)(C+a(A-C))$ for $b \in \mathbb{R}$, ℓ is a line. Since $B+a(A-C), C+a(A-C) \in \ell$ and

$$(B + a(A - C)) - (C + a(A - C)) = B - C,$$

ℓ is parallel to ℓ_{BC}.

2. Exercise 1.18

Solution: Set

$$\frac{1}{6}(A + 2B + 3C) + X\,(aA + bB + cC) = (1 + X)\left(\frac{1}{5}(2A + 2B + C)\right)$$

with $a + b + c = 1$. Comparing the coefficients of A, B, C, we get

$$\begin{aligned}\frac{1}{6} + aX &= \frac{2}{5}(X + 1),\\ \frac{1}{3} + bX &= \frac{2}{5}(X + 1),\\ \frac{1}{2} + cX &= \frac{1}{5}(X + 1).\end{aligned}$$

Thus,

$$(a - 2/5)X = \frac{7}{30}, \quad (b - 2/5)X = \frac{1}{15}, \quad (c - 1/5)X = -\frac{3}{10}.$$

Since $X > 0$, a and b cannot be zero. Thus, $c = 0$, $X = 3/2$, $a = 5/9$, and $b = 4/9$.

3. Exercise 1.20

Solution: Since

$$\begin{aligned}G - A &= \frac{b}{a+b+c}(B - A) + \frac{c}{a+b+c}(C - A),\\ A' - A &= \frac{b}{b+c}(B - A) + \frac{c}{b+c}(C - A),\end{aligned}$$

we see that

$$\frac{G - A}{A' - A} = \frac{b + c}{a + b + c}.$$

Similarly,

$$\frac{G - B}{B' - B} = \frac{c + a}{a + b + c}, \quad \frac{G - C}{C' - C} = \frac{a + b}{a + b + c}.$$

4. Exercise 1.21

Solution: Since

$$G - A = \frac{b}{a+b+c}(B-A) + \frac{c}{a+b+c}(C-A),$$
$$G - A' = (G-A) - (A'-A) = -\frac{b}{a+b+c}\frac{a}{b+c}(B-A) - \frac{c}{a+b+c}\frac{a}{b+c}(C-A),$$

we have

$$\frac{G-A}{G-A'} = -\frac{b+c}{a} = 1 - \frac{a+b+c}{a}.$$

By symmetry and arithmetic–geometric inequality, we get

$$\begin{aligned}\frac{G-A}{G-A'} + \frac{G-B}{G-B'} + \frac{G-C}{G-C'} &= 3 - (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ &= -\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) - \left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right) \\ &\leq -6\end{aligned}$$

and equality holds if and only if $a = b = c$. Thus, G is the centroid of $\triangle ABC$.

5. Let $A = (1,1), B = (3,0), C = (4,5)$. Let

$$A' = \frac{1}{3}B + \frac{2}{3}C, \quad B' = \frac{4}{9}C + \frac{5}{9}A,$$

and C' be a point on the line joining A and B. Suppose $\ell_{AA'}$, $\ell_{BB'}$, and $\ell_{CC'}$ are concurrent.

(a) Find C'.

(b) Let G be the intersection of $\ell_{AA'}$, $\ell_{BB'}$, and $\ell_{CC'}$. Find the barycentric coordinate of G with respect to A, B, C.

Solution: (a): Let $a = 5$, $b = 2$, and $c = 4$. Then,

$$A' = \frac{bB + cC}{b+c}, \quad B' = \frac{cC + aA}{c+a}$$

Thus, Theorem of Ceva yields

$$C' = \frac{aA + bB}{a+b} = \frac{5A + 2B}{7}.$$

(b): The barycentric coordinate of G is $(5/11, 2/11, 4/11)$.