MATH 403 LECTURE NOTE
WEEK 10

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1. CIRCLES (SEC. 3.3)

Definition 1.1. *The circle with center D and radius r is the set of points X satisfying $|X - D| = r$.*

Theorem 1.2 (Thales)**.** *Consider a triangle $\triangle ABC$. Let C' be the midpoint of A and B. Let $\mathcal{S}$ be the circle with center C' and radius $\frac{1}{2}|\overline{AB}|$. (That is, the line segment $\overline{AB}$ is the diameter of $\mathcal{S}$.) Then, $C \in \mathcal{S}$ if and only if $\overline{AC}$ is perpendicular to $\overline{BC}$.*

Proof. The point C lies on $\mathcal{S}$ if and only if

$$|C - \frac{1}{2}(A+B)| = |\frac{1}{2}(A-B)|,$$
$$|(C-A) + (C-B)| = |(C-A) - (C-B)|,$$
$$(C-A)\cdot(C-B) = 0.$$

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Proposition 1.3. *The image of a circle under a dilatation is a circle. Furthermore, every dilatation preserves the center of a circle.*

Proof. Let $\mathcal{S}$ be the circle with center D and radius r. Then, for $X \in \mathcal{S}$, we have

$$|\tau_A(X) - \tau_A(D)| = |X - D| = r,$$
$$|\delta_{C,s}(X) - \delta_{C,s}(D)| = |s(X-D)| = |s|r.$$

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Theorem 1.4 (Nine-point circle theorem)**.** *Consider a triangle $\triangle ABC$. Let A', B', C' be the midpoints of A, B, C, H the orthocenter, and*

$$A'' = \frac{1}{2}(A+H), \quad B'' = \frac{1}{2}(B+H), \quad C'' = \frac{1}{2}(C+H).$$

Let D, E, F be the feet of the altitudes ℓ_A, ℓ_B, ℓ_C. Then, there exists a circle $\mathcal{S}$ that contains the nine points, $A', B', C', A'', B'', C'', D, E, F$. Furthermore, the center of $\mathcal{S}$ is given by

$$N = \frac{1}{2}(H+K)$$

where K is the circumcenter of $\triangle ABC$. This circle is called the Euler circle *or the* Feuerbach circle*.*

Proof. Let K be the circumcenter and $\mathcal{S}$ be the circumcircle (with center K and passing through A, B, C). Let $\alpha = \delta_{G,-\frac{1}{2}}$, where G is the centroid. Note that $\alpha(A) = A'$, $\alpha(B) = B'$, $\alpha(C) = C'$. Since every dilatation sends a circle to a circle, the image $\mathcal{S}' = \alpha(\mathcal{S})$ is the circle passing through A', B', C' with center $N := \alpha(K)$.

Let ℓ_A be the altitude of A and ℓ_{BC} the perpendicular bisector of $\overline{BC}$. Then, one can see that the image $\alpha(\ell_A)$ is ℓ_{BC}. Let H be the orthocenter, then $\alpha(H) = D$. Note that $N = \alpha(K) = \alpha^2(H) = \frac{1}{2}(K+H)$. Since D is the foot of ℓ_A and $\ell_A \mathbin{/\!/} \ell_{BC}$, we have $|N - D| = |N - A'|$. To see this, let $X := K - A'$. Since ℓ_A is parallel to ℓ_{BC}, there exists $t \in \mathbb{R}$ such that $H = D + tX$. Since $K = X + A'$, $N = \frac{1}{2}(H+K) = \frac{1}{2}(D+A') + \frac{1+t}{2}X$. Thus, N is on the perpendicular bisector of $\overline{DA'}$ which yields $|N - D| = |N - A'|$. This implies that $D \in \mathcal{S}'$. Similarly, $E, F \in \mathcal{S}'$.

Let $\beta = \delta_{H,\frac{1}{2}}$. Then, $\beta(K) = G$. Since every dilatation preserves the center of a circle, β maps $\mathcal{S}$ to $\mathcal{S}'$. Since $\beta(A) = A'', \beta(B) = B'', \beta(C) = C''$, the proof is complete. ■

2. Cauchy–Schwarz inequality (Sec. 3.4)

Theorem 2.1. *For all $X, Y \in \mathbb{R}^2$, we have*

$$X \cdot Y \leq |X||Y|.$$

The equality holds if and only if $X = rY$ or $Y = rX$ for some r.

Proof. If $X = Y = O$, the statement holds. Suppose $Y \neq O$. Define a function $f(t) = |X - tY|^2$ for $t \in \mathbb{R}$. Since $f(t) \geq 0$ for all t and

$$f(t) = |X|^2 - 2tX \cdot Y + t^2|Y|^2,$$

we have $|X||Y| \geqslant X \cdot Y$.

If $|X||Y| = X \cdot Y$, then there exists t_0 such that $f(t_0) = 0$. Thus, $X = t_0Y$. The converse also holds. ∎

Theorem 2.2 (Triangle inequality). *For any $X, Y \in \mathbb{R}^2$, we have*

$$|X + Y| \leq |X| + |Y|, \quad |X - Y| \geqslant |X| - |Y|.$$

Proof. It follows from the Cauchy–Schwarz inequality that

$$|X + Y|^2 = |X|^2 + 2X \cdot Y + |Y|^2 \leqslant |X|^2 + 2|X||Y| + |Y|^2 = (|X| + |Y|)^2.$$

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Recall that the distance between X and Y is defined by

$$d(X, Y) := |X - Y|.$$

The distance satisfies the following properties.

Proposition 2.3. *Let $X, Y, Z \in \mathbb{R}^2$.*

(1) $d(X, Y) = d(Y, X)$.
(2) $d(X, Y) \geqslant 0$. *The distance equals to zero if and only if $X = Y$.*
(3) $d(X, Z) \leq d(X, Y) + d(Y, Z)$.

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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