MATH 403 LECTURE NOTE
WEEK 15

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1. Central reflection (Sec. 4.4)

We have seen that if an isometry α has two distinct fixed points of α, then every point on the line ℓ determined by the fixed points is a fixed point. If there is another fixed point away from the line ℓ, then α should be the identity. Otherwise, we have seen that α is a reflection in ℓ. That is, every isometry having two distinct fixed points is either identity or a reflection. Now, we consider isometries with a unique fixed point.

Recall that a central reflection σ_C is defined by $\sigma_C(X) = 2C - X$. We have seen that it is an isometry.

Lemma 1.1. *Let α be an involutive isometry and $X \in \mathbb{R}^2$. Let M be the midpoint of X and $\alpha(X)$, then $\alpha(M) = M$.*

Proof. If X is a fixed point, the statement is true. Supppose X is not a fixed point. Since M is the midpoint, $|M - \alpha(X)| = |M - X|$. Since α is an involutive isometry,

$$|M - \alpha(X)| = |\alpha(M) - X| = |\alpha(M) - \alpha(X)| = |M - X|.$$

Thus, $\alpha(M)$ belongs to the perpendicular bisector of X and $\alpha(X)$. Since α maps $\ell_{X\alpha(X)}$ to itself and $M \in \ell_{X\alpha(X)}$, $\alpha(M)$ belongs to the same line. Thus, $\alpha(M)$ belongs to the intersection of $\ell_{X\alpha(X)}$ and the perpendicular bisector, which implies $\alpha(M) = M$. ∎

Second Proof. It follows from $\alpha(aA + bB) = a\alpha(A) + b\alpha(B)$ for $a + b = 1$ that

$$\alpha(M) = \alpha((X + \alpha(X)/2)) = \frac{1}{2}\alpha(X) + \frac{1}{2}\alpha(\alpha(X)) = M.$$

∎

Theorem 1.2. *Let α be an involutive isometry with a unique fixed point C. Then $\alpha = \sigma_C$.*

Proof. Let $X \neq C$, then $\alpha(X) \neq X$. By the lemma, the midpoint is a fixed point. Since C is the unique fixed point, $C = \frac{1}{2}(X + \alpha(X))$, which yields $\alpha = \sigma_C$. ∎

Proposition 1.3. *Let α be an isometry. Then, $\alpha\sigma_C\alpha^{-1} = \sigma_{\alpha(C)}$.*

2. Unique Fixed Point (Sec. 4.5)

Theorem 2.1. *Let α be an isometry with a unique fixed point P. Then, $\alpha = \sigma_{\ell_1}\sigma_{\ell_2}$ for lines ℓ_1 and ℓ_2 with $P \in \ell_1 \cap \ell_2$.*

Proof. Let $X \neq P$, then $|X - P| = |\alpha(X) - \alpha(P)| = |\alpha(X) - P|$. Thus, P is on the perpendicular bisector ℓ of X and $\alpha(X)$. Consider $\beta = \sigma_\ell\alpha$, then $\beta(X) = X$ and $\beta(P) = P$. It follows from the priveous theorem that either β is the identity or a reflection. If β is the identity, then $\alpha = \sigma_\ell$ so that P is not a unique fixed point, which contradicts to the assumption. Thus, α is the composition of two reflections. ∎

Corollary 2.2. *Let α be an isometry with at least one fixed point. Then, α is either a reflection or the composition of two reflections.*

Theorem 2.3. *Every isometry can be written as the product of not more than three reflections.*

Proof. It suffices to consider the case where there is no fixed point. Let $P \in \mathbb{R}^2$. Then $Q = \alpha(P) \neq P$. Let ℓ be the perpendicular bisector of $\overline{PQ}$. Then, $\sigma_\ell\alpha$ has at least one fixed point P. Thus, $\sigma_\ell\alpha$ is either a reflection or the composition of two reflection. ∎

3. Compositions of Involutions (Sec. 4.6)

Let α, β be involutions. Is the composition also an involution? That is,

$$(\alpha\beta)^2 = \alpha\beta\alpha\beta = \mathrm{Id}?$$

This is equivalent to $\alpha\beta = \beta\alpha$. We say two maps α, β commute if $\alpha\beta = \alpha\beta$. We consider the conditions when involutions commute. In particular, we consider reflections and central reflections.

Proposition 3.1. *Let σ_1, σ_2 be reflections in distinct lines ℓ_1, ℓ_2. Then the maps commute if and only if the lines are perpendicular. In this case, $\sigma_1\sigma_2 = \sigma_C$ where $C \in \ell_1 \cap \ell_2$.*

Proof. We have seen that $\sigma_1\sigma_2\sigma_1 = \sigma_{\sigma_1(\ell_2)}$. Thus, $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if and only if $\sigma_1(\ell_2) = \ell_2$. Since $\ell_1 \neq \ell_2$, it is equivalent to $\ell_1 \perp \ell_2$ as desired. The second assertion is an exercise. ∎

Proposition 3.2. *A central reflection σ_C and a reflection σ_ℓ commute if and only if $C \in \ell$.*

Proof. Since $\sigma_\ell\sigma_C\sigma_\ell = \sigma_{\sigma_\ell(C)}$, they commute if and only if $\sigma_\ell(C) = C$. ∎

4. Translations

Theorem 4.1. *The composition of two reflections in parallel lines is a translation.*

Proof. Let σ_1, σ_2 be reflections in lines $\ell_1 \mathbin{/\mkern-5mu/} \ell_2$. Let ℓ be a line perpendicular to ℓ_1, ℓ_2 and $N \in \ell \cap \ell_1$, $M \in \ell \cap \ell_2$. We have seen that $\sigma_1\sigma_\ell = \sigma_N$ and $\sigma_\ell\sigma_2 = \sigma_M$. Thus,

$$\sigma_1\sigma_2 = \sigma_N\sigma_M = \tau_A, \quad A = 2(N - M).$$

∎

Remark 4.2. *It follows from the proof that every translation can be written as the product ot two parallel reflections, or the product of central reflections. Consider the product of three parallel reflections. Then, it is the product of a translation and a reflection, which is another reflection in a parallel line.*

5. Rotations

We consider a rotation with center $C \in \mathbb{R}^2$ through the oriented angle $\theta \in (-\pi, \pi]$. We define a map $\rho_{C,\theta}$ by $\rho_{C,\theta}(X) = X'$ with

$$|X - C| = |X' - C|, \quad \measuredangle(X - C, X' - C) = \theta.$$

Theorem 5.1. *A rotation is an isometry.*

Proof. Let $\alpha = \rho_{C,\theta}$, $X, Y \in \mathbb{R}^2$, and $X' = \alpha(X), Y' = \alpha(Y)$. Then,

$$\begin{aligned}
|X' - Y'|^2 &= |(X' - C) - (Y' - C)|^2 \\
&= |X' - C|^2 - 2|X' - C||Y' - C| \cos\measuredangle(X' - C, Y' - C) + |Y' - C|^2 \\
&= |X - C|^2 - 2|X - C||Y - C| \cos\measuredangle(X - C, Y - C) + |Y - C|^2 \\
&= |X - Y|^2.
\end{aligned}$$

∎

We define the oriented angle between two lines ℓ_{AB}, ℓ_{CD} by $\measuredangle(A - B, C - D)$. We use the notation $\measuredangle(\ell_{AB}, \ell_{CD})$.

Theorem 5.2. *Let ℓ_1, ℓ_2 be lines with $C \in \ell_1 \cap \ell_2$ and $\theta = 2\measuredangle(\ell_1, \ell_2)$. Then, $\rho_{C,\theta} = \sigma_{\ell_2}\sigma_{\ell_1}$.*

Proof. Let $\alpha = \rho_{C,\theta}\sigma_{\ell_1}\sigma_{\ell_2}$. Note that C is a fixed point of α. Consider a circle $\mathcal{S}$ with center C and radius 1. Let $N \in \mathcal{S} \cap \ell_1$ and $M \in \mathcal{S} \cap \ell_2$. Let $L = \sigma_{\ell_2}(N)$. We claim that M, L are fixed point of α. (Exercise) Since α fixes three non-collinear points, it should be the identity, which completes the proof. ∎

Theorem 5.3. *Let ℓ_i, $i = 1, 2, 3$ be concurrent lines with concurrent point C. Then, $\sigma_1\sigma_2\sigma_3$ is a reflection in a line ℓ_4 passing through C.*

Theorem 5.4. *(1) An isometry with a unique fixed point C is a rotation with center C.*
(2) An isometry with at least one fixed point is one of the identity, a rotation, and a reflection.

Theorem 5.5. *Let $\alpha = \rho_{C,\theta}$ be such that $\alpha \neq \mathrm{Id}$ and $\alpha(\ell) = \ell$. Then, $\alpha = \sigma_C$.*

Proof. Exercise. ∎

Theorem 5.6. *The composition of two rotations is either a rotation or a translation.*

6. Glide Reflections

A glide reflection in a direction $T \in \mathbb{R}^2 \setminus \{O\}$ is a composition $\tau_T \sigma_\ell$, where ℓ is a line parallel to T.

Proposition 6.1. *Let T be a nonzero vector and ℓ a line. Then ℓ is parallel to T if and only if $\tau_T \sigma_\ell = \sigma_\ell \tau_T$.*

Proof. Since $\tau_T \sigma_\ell \tau_T^{-1} = \sigma_{\tau_T(\ell)}$, τ_T and σ_ℓ commute if and only if $\tau_T(\ell) = \ell$. ∎

Note that a glide reflection has no fixed point.

7. Classification of Isometries

Theorem 7.1. *Every isometry is either a reflection, or a translation, or a rotation, or a glide reflection.*

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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