MATH 403 LECTURE NOTE
WEEK 13

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1. Reflection (Sec. 4.3)

Let ℓ be a line in the plane. We consider the reflection map σ_ℓ in a line ℓ. If P is on ℓ, then P should be a fixed point of the map. Suppose $P \notin \ell$ and σ_ℓ maps P to P'. Then, ℓ is the perpendicular bisector of the line segment $\overline{PP'}$.

Suppose $O \in \ell$ and $Y \in \ell$ with $|Y| = 1$. For any $X \in \mathbb{R}^2$, we have

$$\sigma_\ell(X) = X + 2(\mathrm{Proj}_Y X - X) = 2\mathrm{Proj}_Y X - X = 2(X \cdot Y)Y - X.$$

For general line ℓ (not necessarily passing through the origin), we find a vector R such that $\tau_R(\ell)$ passes through O. Then, we will see that

$$\sigma_\ell = \tau_R^{-1}\sigma_{\tau_R(\ell)}\tau_R.$$

In what follows, we focus on the case where $O \in \ell$.

Definition 1.1. *A bijection map α is called an involution if it is not the identity map and $\alpha^2 = \mathrm{Id}$.*

Proposition 1.2. *Every reflection is an involutive isometry.*

Proof. It suffices to consider a line ℓ with $O, Y \in \ell$ and $|Y| = 1$. Consider a parallelogram generated by X and $\sigma_\ell(X)$. Since the diagonals of the parallelogram are perpendicular, it is a rhombus. This implies that $|\sigma_\ell(X)| = |X|$ for all X. For $X, Z \in \mathbb{R}^2$, we have

$$\begin{aligned}
|\sigma_\ell(X) - \sigma_\ell(Z)| &= |2(X \cdot Y)Y - X - 2(Z \cdot Y)Y + Z| \\
&= |2\mathrm{Proj}_Y(X - Z) - (X - Z)| \\
&= |X - Z|.
\end{aligned}$$

Thus, σ_ℓ is an isometry. We also have

$$\begin{aligned}
\sigma_\ell^2(X) &= \sigma_\ell(2(X \cdot Y)Y - X) \\
&= 2[(2(X \cdot Y)Y - X) \cdot Y]Y - (2(X \cdot Y)Y - X) \\
&= 2(X \cdot Y)Y - (2(X \cdot Y)Y - X) \\
&= X
\end{aligned}$$

for all X. Thus, σ_ℓ is an involution. ∎

Theorem 1.3. *Let α be an isometry with $\alpha \neq \mathrm{Id}$. Suppose there exist two distinct fixed points P, Q. Then, α is a reflection in the line ℓ_{PQ}.*

Proof. If $R \in \ell_{PQ}$, then we have seen that $\alpha(R) = R$. Let $R \notin \ell_{PQ}$. If $\alpha(R) = R$, then α should be the identity map. Let $S = \alpha(R) \neq R$. Since α is an isometry, we have

$$\begin{aligned}
|P - S| = |\alpha(P) - \alpha(R)| = |P - R|, \\
|Q - S| = |\alpha(Q) - \alpha(R)| = |Q - R|.
\end{aligned}$$

This means that P and Q are on the perpendicular bisector of $\overline{RS}$. That is, $S = \alpha(R) = \sigma_{\ell_{PQ}}(R)$ for all R as desired. ∎

Corollary 1.4. *Let α be an isometry. Suppose that α is an involution and fixes a line ℓ. Then, $\alpha = \sigma_\ell$.*

Proposition 1.5. *Let α be an isometry, then $\sigma_{\alpha(\ell)} = \alpha\sigma_\ell\alpha^{-1}$.*

Proof. It suffices to show $\alpha\sigma_\ell\alpha^{-1}$ is an involution and fixes $\alpha(\ell)$. (Exercise) ∎

REFERENCES

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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