MATH 403 LECTURE NOTE
WEEK 3

DAESUNG KIM

1. Baricentric coordinates (Sec 1.11)

Theorem 1.1. *Let $A, B, C \in \mathbb{R}^2$ be points which do not lie on a line. For every P in the plane, there is a unique representation*

$$P = aA + bB + cC$$

with $a + b + c = 1$.

Proof. Subtracting A on the both sides, we have

$$P - A = (aA + bB + cC) - (aA + bA + cA) = b(B - A) + c(C - A).$$

Here, I used the assumption that $a + b + c = 1$.

Recall that if X and Y are vectors such that $\ell_{OX} \neq \ell_{OY}$, then for every $Z \in \mathbb{R}^2$, there is a unique representation $Z = rX + sY$.

Here, we assume that A, B, C are not on the same line. In other words, if we let $X = B - A$ and $Y = C - A$, then $\ell_{OX} \neq \ell_{OY}$. Thus, there is a unique representation for $Z := P - A$ in terms of X and Y. That is,

$$P - A = b(B - A) + c(C - A)$$

for some $b, c \in \mathbb{R}$. Thus, we have a unique representation

$$P = (1 - b - c)A + bB + cC = aA + bB + cC.$$

∎

Definition 1.2. *Let $A, B, C \in \mathbb{R}^2$ be points which do not lie on a line. The barycentric coordinate of P with respect to A, B, C is defined by (a, b, c) such that $a, b, c \in \mathbb{R}$ with $a + b + c = 1$ and*

$$P = aA + bB + cC.$$

Remark 1. *Let $A, B, C \in \mathbb{R}^2$ be points which do not lie on a line. Every point P can be also written as*

$$P = \frac{a}{a+b+c}A + \frac{b}{a+b+c}B + \frac{c}{a+b+c}C.$$

Such a, b, c are also unique. Then, P is the centroid of mass-points (a, A), (b, B) and (c, C).

2. Theorem of Ceva (Sec 1.12)

Let $A, B, C \in \mathbb{R}^2$ form a triangle and $a, b, c > 0$. Let

$$A' = \frac{bB + cC}{b + c}, \quad B' = \frac{cC + aA}{c + a}, \quad C' = \frac{aA + bB}{a + b}.$$

Then, we have seen that the lines $\ell_{AA'}$, $\ell_{BB'}$, and $\ell_{CC'}$ are concurrent and

$$\ell_{AA'} \cap \ell_{BB'} \cap \ell_{CC'} = \{G\}$$

where G is the centroid of $(a, A), (b, B), (c, C)$.

Theorem 2.1. *Let $A, B, C \in \mathbb{R}^2$ form a triangle. Let A', B', C' be points on the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Assume that A' is distinct from B, C, B' from C, A, and C' from A, B. Then, $\ell_{AA'}, \ell_{BB'}, \ell_{CC'}$ are concurrent if and only if*

$$\frac{A'-B}{A'-C} \cdot \frac{B'-C}{B'-A} \cdot \frac{C'-A}{C'-B} = -1. \tag{1}$$

Proof. Suppose that $\ell_{AA'}, \ell_{BB'}, \ell_{CC'}$ are concurrent in a point G. Then, G can be written as $G = aA+bB+cC$ for some $a, b, c \in \mathbb{R}$ with $a+b+c=1$. Then, we have

$$G - A = b(B-A) + c(C-A).$$

Since G lies on the line $\ell_{AA'}$, there exists $r \in \mathbb{R}$ such that $G - A = r(A'-A)$, that is,

$$A' - A = rb(B-A) + rc(C-A).$$

Since A' is on $\overline{BC}$ and distinct from B and C, we know $rb + rc = 1$ and $rb \neq 0$, $rc \neq 0$. Therefore, we have

$$A' = \frac{b}{b+c}B + \frac{c}{b+c}C, \quad \frac{A'-B}{A'-C} = -\frac{c}{b}.$$

Similarly, we get

$$B' = \frac{c}{c+a}C + \frac{a}{c+a}A, \quad \frac{B'-C}{B'-A} = -\frac{a}{c},$$
$$C' = \frac{a}{a+b}A + \frac{b}{a+b}B, \quad \frac{C'-A}{C'-B} = -\frac{b}{a}.$$

Thus, the equation (1) holds.

Suppose that the equation (1) holds. Since A' is on the side $\overline{BC}$, there exists $t \in \mathbb{R}$ such that $A' = (1-t)B + tC$. Since A' is distinct from B and C, we know that $t \in (0,1)$. Let $b = 1$ and $c = t/(1-t)$, then

$$A' = \frac{b}{b+c}B + \frac{c}{b+c}C, \quad \frac{A'-B}{A'-C} = -\frac{c}{b}.$$

Similarly, B' can be written as $B' = (1-s)C + sA$ for some $s \in (0,1)$. Let $a = st/((1-s)(1-t))$, then

$$B' = \frac{c}{c+a}C + \frac{a}{c+a}A, \quad \frac{B'-C}{B'-A} = -\frac{a}{c},$$

By (1), we have $C' - A = -\frac{b}{a}(C'-B)$, which implies

$$C' = \frac{a}{a+b}A + \frac{b}{a+b}B.$$

Then, $\ell_{AA'}$, $\ell_{BB'}$, and $\ell_{CC'}$ are concurrent and

$$\ell_{AA'} \cap \ell_{BB'} \cap \ell_{CC'} = \{G\}$$

where G is the centroid of $(a, A), (b, B), (c, C)$. ∎

Remark 2. *Theorem of Ceva holds for $A' \in \ell_{BC}$, $B' \in \ell_{CA}$, $C' \in \ell_{AB}$ as long as A', B', C' are distinct from A, B, C.*

Theorem 2.2. *Let $A, B, C \in \mathbb{R}^2$ form a triangle and G be a point inside $\triangle ABC$, that is, $G = aA + bB + cC$ with $a + b + c = 1$ and $a, b, c \in (0,1)$. Then, we have*

$$|\triangle GBC| : |\triangle GCA| : |\triangle GAB| = a : b : c.$$

Here, $|\triangle PQR|$ denotes the area of $\triangle PQR$.

Proof. The result follows from

$$\begin{aligned} b : a &= |\triangle CAC'| : |\triangle CC'B| \\ &= |\triangle GAC'| : |\triangle GC'B| \\ &= |\triangle CAC'| - |\triangle GAC'| : |\triangle CC'B| - |\triangle GC'B| \\ &= |\triangle CAG| : |\triangle CGB| \end{aligned}$$

and symmetry. ■

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu