Math 403: Euclidean Geometry

Midterm 2 Solution, Fall 2021

Date: October 20, 2021

1. (24 points) Circle True or False. Do not justify your answer.

   (a) **TRUE** False Let $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}^2$ be maps. If $\alpha \circ \beta$ is one-to-one, then β is one-to-one.

   > **Solution:** Suppose $\beta(X) = \beta(Y)$, then $\alpha\beta(X) = \alpha\beta(Y)$. Since $\alpha\beta$ is one-to-one, we have $X = Y$. Thus, β is also one-to-one.

   (b) True **FALSE** A translation and a central dilatation commute.

   > **Solution:** Let $A \neq O$, then $\tau_A\delta_2(A) = 3A$ and $\delta_2\tau_A(A) = 2A$. Thus, $\tau_A\delta_2 \neq \delta_2\tau_A$.

   (c) **TRUE** False Every central dilatation has at least one fixed point.

   > **Solution:** For any $C \in \mathbb{R}^2$ and $r \in \mathbb{R} \setminus \{0\}$, we have $\delta_{C,r}(C) = C$.

   (d) True **FALSE** For any two lines ℓ_1, ℓ_2, there exists a dilatation α such that $\alpha(\ell_1) = \ell_2$.

   > **Solution:** Every dilatation maps a line to a parallel line. If ℓ_1 is not parallel to ℓ_2, then there is no such map.

   (e) True **FALSE** For any $C, D \in \mathbb{R}^2$ and $r, s \in \mathbb{R} \setminus \{0\}$, the composition $\delta_{C,r} \circ \delta_{D,s}$ is a central dilatation.

   > **Solution:** If $s = 1/r$, then
   >
   > $$\delta_{C,r}\delta_{D,1/r}(X) = (1-r)C + r((1-1/r)D + \frac{1}{r}X) = (1-r)(C-D) + X = \tau_{(1-r)(C-D)}(X).$$

   (f) **TRUE** False Let G be a group. A subset H of G is a group if and only if $ab^{-1} \in H$ for all $a, b \in H$.

   > **Solution:** This was covered in class.

2. (20 points) Give definitions of the following.

   (a) The conjugate of α by μ where $\alpha, \mu : \mathbb{R}^2 \to \mathbb{R}^2$.

Solution: The conjugate $\overline{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\overline{\alpha} = \mu\alpha\mu^{-1}$.

(b) The distance between X and Y, where $X, Y \in \mathbb{R}^2$.

Solution: The distance is defined by $d(X,Y) = |X - Y| = \sqrt{(X-Y)\cdot(X-Y)}$, where $\cdot$ is the scalar product.

(c) An isomorphism between two groups G and H.

Solution: An isomorphism is a map $\varphi : G \to H$ that is a bijection and a homomorphism.

(d) A map is onto.

Solution: Let A, B be sets. A map $f : A \to B$ is onto if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

3. (10 points) Let $C, D \in \mathbb{R}^2$. Show that $\sigma_C \circ \sigma_D$ has a fixed point if and only if $C = D$.

Solution: Suppose $\sigma_C\sigma_D$ has a fixed point. That is, there exists $X \in \mathbb{R}^2$ such that $\sigma_C\sigma_D(X) = X$. Then,

$$\sigma_C\sigma_D(X) = 2C - (2D - X) = 2(C - D) + X = X$$

which implies $C - D = 0$. Thus, $C = D$.

Suppose $C = D$. Since the inverse of σ_C is σ_D. Thus, $\sigma_C\sigma_D = \sigma_C^2 = \mathrm{Id}$ and so every point is a fixed point.

4. (10 points) Let $A \in \mathbb{R}^2$. Answer only one of the following questions.

(a) Is $G = \{\tau_{mA} : m \in \mathbb{N}\}$ with composition a group? Justify your answer.

Solution: Suppose $A \neq O$. Since the inverse of τ_A is $\tau_{-A} \notin G$, G is not a group.
If $A = O$, then $G = \{\mathrm{Id}\}$. Thus, G is a group.

(b) Is $H = \{\delta_{A,r} : r \in \mathbb{R}, r > 0\}$ with composition a group? Justify your answer.

Solution: It suffices to show that $\alpha\beta^{-1} \in H$ for all $\alpha, \beta \in H$. Let $\alpha = \delta_{A,r}$ and $\beta = \delta_{A,s}$ for $r, s \in \mathbb{R}$ with $r, s > 0$. Then,

$$\alpha\beta^{-1} = \delta_{A,r}(\delta_{A,s})^{-1} = \delta_{A,r}\delta_{A,1/s} = \delta_{A,r/s}.$$

Since $r/s \in \mathbb{R}$ and $r/s > 0$, we have $\alpha\beta^{-1} = \delta_{A,r/s} \in H$ as desired.

5. Let α be a dilatation and $A, B \in \mathbb{R}^2$ with $A \neq B$.

(a) (10 points) Show that $\alpha(\ell_{AB}) = \ell_{\alpha(A)\alpha(B)}$.

(b) (10 points) Show that ℓ_{AB} is parallel to $\ell_{\alpha(A)\alpha(B)}$.

Solution:

(a) Suppose $X \in \alpha(\ell_{AB})$, then there exists $Y \in \ell_{AB}$ such that $X = \alpha(Y)$. We know that Y can be written as $Y = (1-r)A + rB$. If $\alpha = \tau_R$ is a translation, then

$$X = \alpha(Y) = \tau_R((1-r)A + rB) = (1-r)\alpha(A) + r\alpha(B) \in \ell_{\alpha(A)\alpha(B)}.$$

If $\alpha = \delta_{C,s}$ is a central dilatation, then

$$X = \alpha(Y) = \delta_{C,s}((1-r)A + rB) = (1-r)\alpha(A) + r\alpha(B) \in \ell_{\alpha(A)\alpha(B)}.$$

Thus, $\alpha(\ell_{AB}) \subseteq \ell_{\alpha(A)\alpha(B)}$.

Suppose $X \in \ell_{\alpha(A)\alpha(B)}$, then there exists r such that $X = (1-r)\alpha(A) + r\alpha(B)$. Note that $\alpha(A) \neq \alpha(B)$ because α is bijective and $A \neq B$. By the same calculation as above for translations and central dilatations, we have

$$X = \alpha((1-r)A + rB).$$

Since $(1-r)A + rB \in \ell_{AB}$, we have $X \in \alpha(\ell_{AB})$. We conclude that $\ell_{\alpha(A)\alpha(B)} = \alpha(\ell_{AB})$.

(b) If $\alpha = \tau_R$ is a translation, then

$$\alpha(X) - \alpha(Y) = (R + X) - (R + Y) = X - Y.$$

If $\alpha = \delta_{C,s}$ is a central dilatation, then

$$\alpha(X) - \alpha(Y) = ((1-s)C + sX) - ((1-s)C + sY) = s(X - Y).$$

Thus, if α is a dilatation, then $\alpha(X) - \alpha(Y) = t(X - Y)$ for some t, which implies that two lines are parallel.

6. Let $\alpha = \delta_{A,2}$. Suppose A, B, C form a triangle and G is the centroid.

(a) (8 points) Show that $B, G, C, \alpha(G)$ form a parallelogram.

(b) (8 points) Let $A' = \frac{1}{2}(B + C)$. Using (a), show that A, G, A' are collinear.

Solution:

(a) Since $G = \frac{1}{3}(A + B + C)$, we have

$$G + \alpha(G) = G + ((1-2)A + 2G) = 3G - A = B + C,$$

which proves (a).

(b) Since $\alpha(G) = (1-2)A + 2G$, $A, G, \alpha(G)$ are collinear. Since the midpoint of G and $\alpha(G)$ is A', we conclude that A, G, A' are collinear.