# Math 403: Euclidean Geometry

## Midterm 3 Solution, Fall 2021

### Date: November 17, 2021

1. (24 points) Circle True or False. Do not justify your answer.

(a) True **FALSE** Let $X, Y \in \mathbb{R}^2$. If $X \cdot Y = 0$, then $X = O$ or $Y = O$.

> **Solution:** If $X = (1, 0)$ and $Y = (0, 1)$, then $X \cdot Y = 0$.

(b) **TRUE** False Let $X, Y, Z \in \mathbb{R}^2$. If $X \cdot Y = Y \cdot Z = Z \cdot X = 0$, then at least one of $X, Y, Z$ is zero.

> **Solution:** Suppose $Y, Z \neq O$. Since $Y \cdot Z = 0$, $O, Y, Z$ are not collinear. Thus, there exists $a, b \in \mathbb{R}$ such that $X = aY + bZ$. Then,
>
> $$X \cdot X = (aY + bZ) \cdot X = aX \cdot Y + bZ \cdot X = 0.$$

(c) True **FALSE** There exist $X, Y \in \mathbb{R}^2$ such that $|X| = 4$, $|X - Y| = 5$, and $|Y| = 10$.

> **Solution:** By the triangle inequality, we have $10 = |Y| \leqslant |X| + |X - Y| = 9$, which is a contradiction.

(d) **TRUE** False For any non-collinear distinct points $A, B, C \in \mathbb{R}^2$, there exists a circle $\mathcal{S}$ such that $A, B, C \in \mathcal{S}$.

> **Solution:** If $K$ is the circumcenter, then $|K - A| = |K - B| = |K - C|$ so there exists such a circle.

(e) True **FALSE** Every isometry maps a line to a parallel line.

> **Solution:** No. If $\alpha((x, y)) = (y, x)$, then it is an isometry and maps the line $x = 0$ to the line $y = 0$.

(f) **TRUE** False Let $A, B, C$ form a triangle, $H$ the orthocenter, $K$ the circumcenter, and $D, E, F$ be the feet of the altitudes of $A, B, C$. Then, the nine point circle theorem tells us that if $N$ is the midpoint of $H$ and $K$, then

$$|D - N| = |E - N| = |F - N|.$$

> **Solution:** The circle in the nine point circle theorem has the center at $N$ and contains the feet $D, E, F$.

2. (25 points) Give definitions of the following.
   (a) Rhombus

   **Solution:** A rhombus is a parallelogram with sides of equal length.

   (b) Altitudes of a triangle

   **Solution:** The altitude $\ell_C$ of a triangle $\triangle ABC$ through $C$ is the line perpendicular to $\ell_{AB}$ through $C$.

   (c) The circumcenter of a triangle

   **Solution:** The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is called the circumcenter.

   (d) The length of a vector $X \in \mathbb{R}^2$

   **Solution:** The length of $X$ is defined by $|X| = \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2}$.

   (e) Orthogonal projection of a vector $Y$ onto a vector $X$

   **Solution:** $\text{Proj}_X Y = \frac{X \cdot Y}{|X|^2} X$.

3. (10 points) Let $X, Y \in \mathbb{R}^2$. Show that $|X + Y| = |X - Y|$ if and only if $X \cdot Y = 0$.

   **Solution:** It follows from $|X + Y|^2 - |X - Y|^2 = 4X \cdot Y$.

4. (a) (5 points) Write down the statement of the Cauchy–Schwarz inequality.
   (b) (8 points) Let $x, y \in \mathbb{R}$ with $x^2 + y^2 = 4$. Find the maximum of $3x + 4y$ and determine for which values $x, y$ the maximum is attained.
   (Hint: Consider $A = (3, 4)$ and $B = (x, y)$ and use the Cauchy–Schwarz inequality.)

   **Solution:**

   (a) For all $X, Y \in \mathbb{R}^2$, we have

   $$X \cdot Y \leq |X||Y|.$$

   The equality holds if and only if $X = rY$ or $Y = rX$ for some $r \geqslant 0$.

   (b) By Cauchy–Schwarz inequality, we have

   $$3x + 4y = A \cdot B \leqslant |A||B| = 10.$$

   Thus, the maximum is 10 and it is attained when $(x, y) = B = rA = (3r, 4r)$ for some $r \geqslant 0$. Since $x^2 + y^2 = 4 = 25r^2$, we have $r = 2/5 \geqslant 0$. Thus, the maximum occurs when $(x, y) = (6/5, 8/5)$.

5. (a) (8 points) Let $A, B \in \mathbb{R}^2$ with $A \neq B$. Show that $X$ is on the perpendicular bisector of $\overline{AB}$ if and only if $|X - A| = |X - B|$.

   (b) (8 points) Let $A, B, C \in \mathbb{R}^2$ form a triangle and $K$ be the circumcenter of $\triangle ABC$. Show that $|K-A| = |K - B| = |K - C|$.

**Solution:**

(a) Let $\ell$ be the perpendicular bisector. Suppose $X \in \ell$. Then

$$(X - \frac{1}{2}(A + B)) \cdot (A - B) = 0.$$

Let $Y = X - A$ and $Z = X - B$, then it is equivalent to $(Y - Z) \cdot (Y + Z) = 0$. This implies that $|Y| = |X - A| = |X - B| = |Z|$.

On the other hand, assume $|Y| = |X - A| = |X - B| = |Z|$. Then, $|Y|^2 - |Z|^2 = 0$ implies $(Y + Z) \cdot (Y - Z) = 0$. Thus, $X \in \ell$.

(b) Since $K$ is on the perpendicular bisector of $\overline{AB}$, Part (a) yields $|K - A| = |K - B|$. Also, $K$ is on the perpendicular bisector of $\overline{BC}$ so that $|K - B| = |K - C|$, which finishes the proof.

6. (12 points) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$. Show that $\alpha$ is an isometry if and only if $(\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)) = (X - Z) \cdot (Y - Z)$ for all $X, Y, Z$.

**Solution:** Suppose $\alpha$ is an isometry. Then,

$$\begin{aligned}
|\alpha(X) - \alpha(Y)|^2 &= |(\alpha(X) - \alpha(Z)) - (\alpha(Y) - \alpha(Z))|^2 \\
&= |\alpha(X) - \alpha(Z)|^2 - 2(\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)) + |\alpha(Y) - \alpha(Z)|^2 \\
&= |X - Z|^2 - 2(\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)) + |Y - Z|^2 \\
&= |X - Y|^2 + 2(X - Z) \cdot (Y - Z) - 2(\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)) + |Y - Z|^2.
\end{aligned}$$

Since $|\alpha(X) - \alpha(Y)| = |X - Y|$, we conclude that

$$(X - Z) \cdot (Y - Z) = (\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)).$$

On the other hand, if $X = Y$, then the assumption implies

$$|X - Z|^2 = (X - Z) \cdot (X - Z) = (\alpha(X) - \alpha(Z)) \cdot (\alpha(X) - \alpha(Z)) = |\alpha(X) - \alpha(Z)|^2$$

for all $X, Z$. Thus, $\alpha$ is an isometry.