MATH 403 LECTURE NOTE
WEEK 8

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1. GROUP OF TRASFORMATIONS (SEC. 2.5)

Let V be a set and G be the set of all bijections (or trasformations, or permutations) $f : V \to V$. Then G with composition forms a group. The group G and its subgroups are called groups of permutations (or trasformations) of V.

Theorem 1.1. *Let $\mathcal{T}$ be the set of all translations $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$, $A \in \mathbb{R}^2$. Then, $\mathcal{T}$ with compositions is a group.*

Proof. By the lemma, it suffices to show that for any $A, B \in \mathbb{R}^2$, $\tau_A \tau_B^{-1} \in \mathcal{T}$. ■

Theorem 1.2. *Let $C \in \mathbb{R}^2$ be fixed and $\mathcal{C}$ be the set of all central dilatations $\delta_{C,r}$ with the fixed center C. Then, $\mathcal{C}$ with compositions is a group.*

Definition 1.3. *We call a bijection map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ a dilatation if α is either a translation or a central dilatation.*

Definition 1.4. *We call a bijection map collineation if it maps a line to a parallel line.*

As we have seen before, dilatations are collineation.

Theorem 1.5. *The set of all dilatations $\mathcal{D}$ with compositions forms a group.*

Proof. Since $(\tau_A)^{-1} = \tau_{-A}$ and $(\delta_{C,r})^{-1} = \delta_{C,1/r}$, it suffices to show that
(1) $\tau_A \tau_B \in \mathcal{D}$,
(2) $\tau_A \delta_{C,r} \in \mathcal{D}$,
(3) $\delta_{C,r} \tau_A \in \mathcal{D}$,
(4) $\delta_{C,r} \delta_{D,s} \in \mathcal{D}$.

We know that (1) is trivial and (3) follows from (2). Thus, it is enough to show (2) and (4). We first have

$$\begin{aligned}\tau_A \delta_{C,r}(X) &= \tau_A((1-r)C + rX) \\ &= (1-r)(\frac{1}{1-r}A + C) + rX \\ &= \delta_{P,r}(X)\end{aligned}$$

where $P = \frac{1}{1-r}A + C$. Also,

$$\delta_{C,r}\delta_{D,s}(X) = (1-r)C + r(1-s)D + rsX = \begin{cases} \tau_{(1-r)(C-D)}(X), & rs = 1, \\ \delta_{Q,rs}(X), & rs \neq 1, \end{cases}$$

where $Q = (1-rs)^{-1}((1-r)C + r(1-s)D)$. ■

2. EQUIVALENCE RELATIONS

Definition 2.1. *A property is said to be invariant under G is it still holds after the transformations of G are applied. A geometric figure is invariant under G if it is mapped to itself by the transformations of G.*

Example 2.2. *Consider G be the set of all dilatations from $\mathbb{R}^2$ to $\mathbb{R}^2$. A line is invariant under G, and a property that a line is parallel is also invariant under G.*

Definition 2.3. *Let G be a group. Two figures F_1 and F_2 in the plane are related by G if there exists α such that $\alpha(F_1) = F_2$. We use the notation $F_1 \sim F_2$.*

Proposition 2.4. *(1) (Transitivity) If $F_1 \sim F_2$ and $F_2 \sim F_3$, then $F_1 \sim F_3$.*

(2) *(Symmetry) If* $F_1 \sim F_2$*, then* $F_2 \sim F_1$.
(3) *(Reflexivity)* $F_1 \sim F_1$.

If a relation $\sim$ satisfies the above three properties, then we say the relation is equivalent relation.

3. Scalar products and orthogonality (Sec. 3.1–2)

Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. The scalar product of X and Y is defined by

$$X \cdot Y = x_1 y_1 + x_2 y_2.$$

The length of X is defined by $|X| = \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2}$. The distance between X and Y is defined by $d(X, Y) = |X - Y|$.

Proposition 3.1. *Let* $X, Y, Z \in \mathbb{R}^2$ *and* $r \in \mathbb{R}$*.*

(1) $X \cdot Y = Y \cdot X$.
(2) $(X + Y) \cdot Z = X \cdot Z + Y \cdot Z$.
(3) $(rX) \cdot Y = r(X \cdot Y)$.
(4) $X \cdot X \geq 0$ *and equality holds if and only if* $X = O$.
(5) $|X + Y|^2 + |X - Y|^2 = 2(|X|^2 + |Y|^2)$.

Proof. (1), (2), (3): Exercise.

(4): Let $X = (x, y)$, then $X \cdot X = x^2 + y^2 = 0$. Since $x^2, y^2 \geqslant 0$ and equal to zero only if $x = 0$, $y = 0$, the proof is complete.

(5): It follows from

$$|X + Y|^2 = (X + Y) \cdot (X + Y) = |X|^2 + |Y|^2 + 2X \cdot Y,$$
$$|X - Y|^2 = (X - Y) \cdot (X - Y) = |X|^2 + |Y|^2 - 2X \cdot Y.$$

■

Two vectors X, Y are said to be orthogonal if $X \cdot Y = 0$.

Example 3.2. *Let* $E_1 = (1, 0)$ *and* $E_2 = (0, 1)$*, then they are orthogonal. In general* $X = (x, y)$ *and* $Y = (-y, x)$ *are orthogonal.*

A rhombus is a parallelogram with sides of equal length.

Proposition 3.3. *A parallelogram is a rhombus if and only if its diagonals are orthogonal.*

Proof. Let A, B, C, D form a parallelogram, then $A + C = B + D$. For notational convenience, we assume $A + C = O = B + D$ without loss of generality. Thus, $C = -A$ and $D = -B$.

If it is a rhombus, then $|A - B| = |A - D| = |A + B|$. Thus,

$$|A - B|^2 = |A|^2 - 2A \cdot B + |B|^2 = |A|^2 + 2A \cdot B + |B|^2 = |A + B|^2,$$

which implies $A \cdot B = 0$. Thus, A and B are orthogonal. Since the diagonals are $2A$ and $2B$, they are orthogonal too. The converse also follows from the same argument. ■

Theorem 3.4. *The scalar product* $X \cdot Y = 0$ *if and only if* $|X - Y|^2 = |X|^2 + |Y|^2$.

Definition 3.5. *A rectangle is a parallelogram with orthogonal sides.*

Proposition 3.6. *A parallelogram is a rectangle if and only if its diagonals have the same length.*

4. Circumcenters and Orthocenters (Sec. 3.2)

Definition 4.1. *The perpendicular bisector* ℓ_n *of the segment* A *and* B *is the line orthogonal to* ℓ_{AB} *and passes through the midpoint of* A *and* B*.*

Proposition 4.2. *Let* ℓ_n *be the perpendicular bisector of* $\overline{AB}$*. Then,* $X \in \ell_n$ *if and only if* $|X - A| = |X - B|$.

Proof. Suppose $X \in \ell_n$, then

$$(X - \frac{1}{2}(A+B)) \cdot (A-B) = 0.$$

Let $A' = X - A$ and $B' = X - B$, then the above equation can be written as $(A' + B') \cdot (A' - B') = 0$, which implies that $|A'| = |B'|$. On the other hand, $|X - A| = |X - B|$ implies the above equation so that X lies on the perpendicular bisector. ■

Theorem 4.3. *The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is called the circumcenter.*

Proof. Let ℓ_1, ℓ_2, ℓ_3 be perpendicular bisectors of the segments BC, CA, and AB respectively. Let $X \in \ell_1 \cap \ell_2$. Then, $|X - A| = |X - B| = |X - C|$. Thus, X should lie on ℓ_3. ■

Definition 4.4. *The altitude ℓ_C of a triangle $\triangle ABC$ through C is the line perpendicular to ℓ_{AB} through C. The intersection point H_C between ℓ_C and ℓ_{AB} is called the foot of ℓ_C.*

Theorem 4.5. *The altitudes of a triangle are concurrent. The point of concurrence is called the orthocenter.*

There are three proofs.

First Proof. This proof relies on the following observation.

Lemma 4.6. *For any points $X, A, B, C \in \mathbb{R}^2$, we have*

$$(X - A) \cdot (B - C) + (X - B) \cdot (C - A) + (X - C) \cdot (A - B) = 0.$$

If $X \in \ell_A \cap \ell_B$, then

$$(X - A) \cdot (B - C) = (X - B) \cdot (C - A) = 0.$$

Thus, we have $(X - C) \cdot (A - B) = 0$, which implies that $X \in \ell_C$.

■

Lemma 4.7. *Consider a triangle $\triangle ABC$. Let A' be the midpoint of B and C. Let ℓ_n be the perpendicular bisector through A', and ℓ_A the altitude through A. Then, $\delta_{G,-2}$ maps ℓ_n to ℓ_A, where G is the centroid.*

Second Proof. Since $\delta_{G,-2}$ maps each perpendicular bisector to the corresponding altitude and the perpendicular bisectors are concurrent, so are the altitudes. Note that it also maps the concurrence point, the circumcenter, to the concurrence point of the altitudes. ■

Third Proof. Consider the image of $\triangle ABC$ under $\delta_{G,-2}$. Then, the image is also a triangle $\triangle PQR$ and A, B, C are the midpoints of P, Q, R. Since the perpendicular bisectors of $\triangle PQR$ coincide with the altitude of $\triangle ABC$, the proof is complete. ■

References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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