Math 403: Euclidean Geometry

Midterm 1 Solution, Fall 2021

Date: September 22, 2021

1. (25 points) Circle True or False. Do not justify your answer.

   (a) True **FALSE** Let A', B', C' be three noncollinear points in the plane. Then, there are at least two triangles $\triangle ABC$ with $\triangle A'B'C'$ as the triangle of midpoints of its sides.

   > **Solution:** For given A', B', C', there is only one such triangle $\triangle ABC$ because
   > $$A = -A' + B' + C', \quad B = A' - B' + C', \quad C = A' + B' - C'.$$

   (b) True **FALSE** If A, B, C are not collinear and $aA + bB + cC = O$, then $a = b = c = 0$.

   > **Solution:** Let $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$. Then, A, B, C are not collinear and $aA + bB + cC = 0$ with $b = c = 0$ and $a = 1$.

   (c) **TRUE** False The point $P = 3B - 2A$ lies on the line ℓ_{AB}.

   > **Solution:** Since $P = (1 - t)A + tB$ with $t = 3$, $P \in \ell_{AB}$.

   (d) **TRUE** False Let $A = (3, -5)$, $B = (2, 1)$, $C = (-4, 2)$, and $D = (-3, -4)$. Then A, B, C, D forms a parallelogram.

   > **Solution:** Since $A + C = (-1, -3) = B + D$, it is a parallelogram.

   (e) True **FALSE** Let A, B, C form a triangle. Menelaus theorem says that the points A', B', C' on the sides of $\triangle ABC$ are collinear if and only if

   $$\frac{A' - B}{A' - C} \cdot \frac{B' - C}{B' - A} \cdot \frac{C' - A}{C' - B} = -1.$$

   > **Solution:** The theorem says A', B', C' are collinear if and only if the product is 1.

2. (a) (5 points) Write down the definition that two lines ℓ_{AB} and ℓ_{CD} are parallel.

   (b) (10 points) Suppose that ℓ_{AB} is parallel to ℓ_{CD}, and ℓ_{CD} is parallel to ℓ_{EF}. Show that ℓ_{AB} is parallel to ℓ_{EF}.

Solution:

(a) Two lines ℓ_{AB} and ℓ_{CD} are parallel if there exists a nonzero $t \in \mathbb{R}$ ($t \neq 0$) such that $A - B = t(C - D)$.

(b) By the assumption, there exist $s, t \in \mathbb{R}$ such that $t \neq 0$, $s \neq 0$, $A - B = t(C - D)$, and $C - D = s(E - F)$. Thus, we have $A - B = ts(E - F)$ and $ts \neq 0$. By definition, ℓ_{AB} is parallel to ℓ_{EF}.

3. (a) (5 points) Write down the definition of a parallelogram.

   (b) (12 points) Let A, B, C, D form a parallelogram. Let M be the centroid of $\triangle BCD$, N be the centroid of $\triangle CDA$, L be the centroid of $\triangle DAB$, P be the centroid of $\triangle ABC$. Show that a quadrilateral $MNLP$ defines a parallelogram.

Solution:

(a) A quadrilateral $ABCD$ is a parallelogram if $A + C = B + D$.

(b) We have

$$M = \frac{1}{3}(B + C + D), \quad N = \frac{1}{3}(A + C + D), \quad L = \frac{1}{3}(A + B + D), \quad P = \frac{1}{3}(A + B + C).$$

Since $A + C = B + D$, we get

$$M + L = \frac{1}{3}(A + B + C + D) + \frac{1}{3}(B + D) = \frac{1}{3}(A + B + C + D) + \frac{1}{3}(A + C) = N + P.$$

4. Let A, B, C be three noncollinear points in $\mathbb{R}^2$.

   (a) (5 points) Write down the definition of the barycentric coordinate with respect to A, B, C.

   (b) (13 points) Let ℓ be the set of all points P such that its barycentric coordiate (a, b, c) satisfies $a = b + c$. Show that ℓ is a line.

Solution:

(a) Every point P in the plane can be uniquely written as $P = aA + bB + cC$ with $a + b + c = 1$. Then (a, b, c) is called the barycentric coordinate of P with respect to A, B, C.

(b) Every point P on the line ℓ can be written as $P = aA + bB + cC$ with $a + b + c = 1$ and $a = b + c$. Note that $a = b + c = \frac{1}{2}$. Thus,

$$P = aA + bB + cC = b(A + B) + c(A + C) = tC' + (1 - t)B'$$

where $t = 2b$, $B' = \frac{1}{2}(A + C)$, and $C' = \frac{1}{2}(A + B)$. Since A, B, C are not collinear, we know that $B' \neq C'$. (Otherwise, $B = C$ so that A, B, C should be collinear.) Thus, $\ell = \ell_{B'C'}$.

5. Let $A, B, C \in \mathbb{R}^2$ form a triangle.

(a) (10 points) Write down the statement of Ceva's theorem.

(b) (15 points) Let

$$A' = \frac{1}{3}(2B + C), \quad B' = \frac{1}{3}(2C + A), \quad C' = \frac{1}{3}(2A + B).$$

Are the three lines $\ell_{AA'}, \ell_{BB'}, \ell_{CC'}$ concurrent? Justify your answer.

Solution:

(a) Let A, B, C form a triangle and A', B', C' lie on the sides of $\triangle ABC$ and be distinct from A, B, C. Then, the lines $\ell_{AA'}, \ell_{BB'}, \ell_{CC'}$ are concurrent if and only if

$$\frac{A' - B}{A' - C} \cdot \frac{B' - C}{B' - A} \cdot \frac{C' - A}{C' - B} = -1.$$

(b) One can see that

$$\frac{A' - B}{A' - C} = \frac{B' - C}{B' - A} = \frac{C' - A}{C' - B} = -\frac{1}{2}.$$

Since the product is $-\frac{1}{8}$, the lines are not concurrent.