# MATH 403 FALL 2021: FINAL EXAM PRACTICE PROBLEMS

## 1. Definitions

(a) Two lines are parallel (perpendicular).
(b) Centroid, barycentric coordinates
(c) Fixed points of a map.
(d) Central dilatation, translation, dilatation, central reflection.
(e) Scalar product of $X, Y$, the length of $X$, the distance between $X, Y$.
(f) Parallelogram, rhombus, rectangle, circle
(g) Perpendicular bisectors, altitude, the foot of altitude, circumcenter, orthocenter
(h) Distance between $X$ and $Y$, The length of $X$
(i) Orthogonal projection, angle, determinant
(j) An isometry, a linear isometry, a linear map
(k) Reflection, involutions, rotations

## 2. Statements of Theorems

(a) Ceva's Theorem, Menelaus' Theorem
(b) Pythagoras Theorem, Thales Theorem, Parallelogram Law
(c) Triangle inequality, Cauchy–Schwarz inequality
(d) Nine point circle theorem

## 3. Examples

Give an example, or explain why no such example exists.

(a) Find an isometry $\alpha$ such that $\alpha(0,0) = (0,1)$ and $\alpha(1,0) = (3,0)$.
(b) Find an isometry $\alpha$ such that $\alpha \neq \mathrm{Id}$, $\alpha(0,0) = (0,0)$, and $\alpha(1,1) = (1,1)$. In this case, what is $\alpha(3,3)$? Can we find two different such isometries?
(c) Find an involutive isometry $\alpha$ such that $\alpha(0,0) = (4,0)$.

## 4. Proof or Disproof

### 4.1. **Lines and Triangles.**

(a) Two distinct parallel lines do not intersect.
(b) Two lines are perpendicular to a line $\ell$, then they are parallel.
(c) The centroid of a triangle is the centroid of the triangle of the midpoints of its sides.
(d) The altitudes (the medians, the perpendicular bisectors) of a triangle are concurrent.
(e) A vector $X$ is on the perpendicular bisector of $\overline{AB}$ if and only if $|X - A| = |X - B|$.

4.2. **Dilatations.**

(a) For a dilatation $\alpha$, $\alpha(\ell_{AB}) = \ell_{\alpha(A)\alpha(B)}$.

(b) For a dilatation $\alpha$, $\ell_{AB} \mathbin{/\!/} \ell_{\alpha(A)\alpha(B)}$.

(c) Every dilatation is an isometry.

(d) Translation preserves the centroid of three points.

(e) Every dilatation has at least one fixed point.

(f) Central dilatations preserve midpoints.

(g) $\delta_{C,r}$ is an involution if $r = 1$ or $r = -1$.

4.3. **Group theory.**

(a) Let $V$ be a set and $G$ be the set of all bijections $\alpha : V \to V$. Show that $G$ with composition is a group.

(b) The set of all translations forms a group.

(c) The set of all central dilatations forms a group.

(d) The set of all isometries forms a group.

(e) For fixed $A \neq O$, $\{\tau_{cA} : c \in \mathbb{R}\}$ is a group.

(f) For fixed $C \in \mathbb{R}^2$, $\{\delta_{C,r} : r \in \mathbb{R}, r > 0\}$ is a group.

4.4. **Scalar Product.**

(a) If $X \cdot Y = Y \cdot Z = Z \cdot X = 0$, then one of $X, Y, Z$ is zero.

(b) If $|X| = 3$, $|Y| = 4$, then $|X \cdot Y| \leqslant 12$.

(c) If $X$ is perpendicular to $Y$, then $|X + Y|^2 = |X|^2 + |Y|^2$.

4.5. **Isometry.**

(a) If $\alpha$ is an isometry, then $(\alpha(X) - \alpha(Z)) \cdot (\alpha(Y) - \alpha(Z)) = (X - Z) \cdot (Y - Z)$ for all $X, Y, Z$.

(b) Every isometry is the composition of a translation and a linear isometry.

(c) If $\alpha$ is an isometry and $a + b + c = 1$, then

$$\alpha(aX + bY + cZ) = a\alpha(X) + b\alpha(Y) + c\alpha(Z).$$

(d) If an isometry has two distinct fixed points, then it is either the identity or a reflection.

(e) If an isometry has a unique fixed point, it is the composition of two reflections.

(f) Let $\tau(x, y) = (x, y + 2)$. Find two lines $\ell, m$ such that $\tau = \sigma_\ell \circ \sigma_m$.

(g) Find a translation and a linear isometry such that $\sigma_C = \tau_R \circ L$.