MATH 403 FALL 2021: HOMEWORK 5 SOLUTION

INSTRUCTOR: DAESUNG KIM
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1. Exercise 2.3

Solution: Let $a' = a/r$ and $b' = b/r$, then $P = aA + bB = a'(rA) + b'(rB)$ with $a' + b' = 1$. Thus, P is a point on the line $\ell_{rA,rB}$. In other words, P lies on the image $\delta_r(\ell_{AB})$.

2. Exercise 2.4(i)

Solution: Let $\alpha = \delta_{A,2}$. Then, $\alpha(B') = C$ and $\alpha(C') = B$. Since α maps from a line to a parallel line, $\alpha(\ell_{BB'})$ is a parallel line through C and $\alpha(\ell_{CC'})$ is a parallel line through B. Also, $\alpha(\ell_{BB'})$ and $\alpha(\ell_{CC'})$ meet at $G' := \alpha(G)$. By definition of α, we know that A, G, G' are collinear. Thus, B, G, C, G' forms a parallelogram. Since the diagonals of a parallelogram bisect each other, ℓ_{BC} and $\ell_{GG'}$ meet at A'. Therefore, G is on the line $\ell_{AA'}$ as desired.

3. Exercise 2.5: Let $\ell_{AA'}$, $\ell_{BB'}$, $\ell_{CC'}$ be three lines concurrent in D. If $\ell_{AB} /\!/ \ell_{A'B'}$, $\ell_{BC} /\!/ \ell_{B'C'}$, then prove $\ell_{AC} /\!/ \ell_{A'C'}$.

Solution: Since D, A, A' are collinear, there exists $r \in \mathbb{R}$ such that $\delta_{D,r}(A) = A'$. Let $\alpha = \delta_{D,r}$. Since $\alpha(\ell_{AB})$ is parallel to ℓ_{AB} and passes through A', we have $B' \in \alpha(\ell_{AB})$. Since D, B, B' are collinear, we get $\alpha(B) = B'$. Similarly, we get $\alpha(C) = C'$. Since $\alpha(\ell_{AC}) = \ell_{A'C'}$, the proof is complete.

4. Exercise 2.6

Solution: Since O, B, B' are collinear, there exists $\alpha = \delta_r$ such that $\alpha(B) = B'$. Similarly, there exists $\beta = \delta_s$ such that $\beta(C) = C'$. Since $X \in \ell_{BB'}$ and $\ell_{B'C} /\!/ \ell_{XC'}$, we know that $\beta(B') = X$. Similarly, $\alpha(C') = Y$. Let $\gamma = \alpha\beta = \beta\alpha = \delta_{rs}$, then $X = \gamma(B)$ and $Y = \gamma(C)$. Thus, $\ell_{BC} /\!/ \ell_{XY} = \gamma(\ell_{BC})$.

5. Exercise 2.8

Solution: It follows from

$$(\beta\alpha)(\alpha^{-1}\beta^{-1}) = \beta(\alpha\alpha^{-1})\beta^{-1} = (\beta e)\beta^{-1} = \beta\beta^{-1} = e$$
$$(\alpha^{-1}\beta^{-1})(\beta\alpha) = \alpha^{-1}(\beta^{-1}\beta)\alpha = (\alpha^{-1}e)\alpha = \alpha^{-1}\alpha = e.$$

6. Exercise 2.9

Solution: Fix A and Q. Since $\sigma_Q^{-1} = \sigma_Q$, the equation $\tau_A = \sigma_Q\sigma_P$ implies $\sigma_P = \sigma_Q\tau_A$. For $X \in \mathbb{R}^2$, we have

$$\sigma_Q\tau_A(X) = 2Q - (A + X) = 2(Q - \frac{1}{2}A) - X = \sigma_P(X)$$

where $P = Q - \frac{1}{2}A$.