# MATH 403 LECTURE NOTE
# WEEK 7

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## 1. Groups (Sec 2.6)

**Definition 1.1.** *Let $G$ be a set equipped with an operation $(x, y) \mapsto xy \in \mathcal{G}$ for all $x, y \in G$. We say $G$ is a group with the operation if*

*(1) There exsits an element $e$ in $G$ such that $ex = xe = x$ for all $x \in G$. We call $e$ the identity.*
*(2) For every $x \in G$, there is an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e$.*
*(3) For all $x, y, z \in G$, we have $x(yz) = (xy)z$.*

**Example 1.2.** *(1) The real numbers $\mathbb{R}$ with addtion is a group.*
*(2) The positive real numbers $\mathbb{R}_+$ with multiplication is a group.*
*(3) The integers $\mathbb{Z}$ with addtion is a group.*
*(4) The even integers $2\mathbb{Z}$ with addtion is a group.*

**Example 1.3.** *(1) The natural numbers $\mathbb{N}$ with addtion is NOT a group.*
*(2) The real numbers $\mathbb{R}$ with multiplication is NOT a group.*
*(3) The odd integers with addtion is NOT a group.*

**Definition 1.4.** *A group $G$ is called commutative or abelian if $xy = yx$ for all $x, y \in G$.*

**Example 1.5.** *Let $A$ be a set and $G$ be the collection of all bijections $f : A \to A$. Then, $G$ with composition is a non-abelian group. (Exercise)*

**Definition 1.6.** *Let $n \in \mathbb{N}$ and $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$. Define the addition and the multiplication on $\mathbb{Z}_n$ by the residues left after division by $n$.*

**Example 1.7.** *One can see that $\mathbb{Z}_n$ with addition is a group.*

**Proposition 1.8.** *Let $G$ be a group and $x \in G$. Then, the identity and the inverse of $x$ are unique.*

**Proposition 1.9.** *Let $G$ be a group and $a, b \in G$.*

*(1) The equation $ax = b$ has a unique solution $x$.*
*(2) The equation $ya = b$ has a unique solution $y$.*

**Proposition 1.10** (Cancellation rules)**.** *Let $\mathcal{G}$ be a group.*

*(1) $ax = ax'$ implies $x = x'$.*
*(2) $ya = y'a$ implies $y = y'$.*

**Example 1.11.** *In $\mathbb{Z}_{12}$, $3x = 5$ has no solution.*

## 2. Subgroups, cyclic groups, and isomorphism

**Definition 2.1.** *Let $G$ be a group and $H$ a subset of $G$. If $H$ is also a group itself, we call $H$ a subgroup of $G$.*

**Example 2.2.** *(1) If $G$ is a group and $e \in G$ is the identity, then $\{e\}$ is a subgroup. This is called the trivial subgroup.*
*(2) The even integers $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.*

**Lemma 2.3.** *Let $G$ be a group and $H$ a subset of $G$. Then $H$ is a subgroup if and only if $ab^{-1} \in H$ for all $a, b \in H$.*

*Proof.* Suppose $H$ is a subgroup, then $ab^{-1} \in H$ for all $a, b \in H$ by definition. Suppose we know that $ab^{-1} \in H$ for all $a, b \in H$. If $a = b$, then $e \in H$. If $a = e$, then every element in $H$ has the inverse in $H$. The associativity works in $H$. For any $a, b \in H$, $ab = a(b^{-1})^{-1} \in H$. Thus $H$ is a group. ■

For an element $x$ in a group $G$, we use the notation

$$x^m := x \cdot x \cdots x \quad (m\text{times})$$

for $m \in \mathbb{N}$. If $m = 0$, $x^m := E$. If $m < 0$, then $x^m = (x^{-1})^m$.

**Definition 2.4.** *Let $G$ a group and $x \in G$. The cyclic group $\langle x \rangle$ generated by $x$ is a subgroup of $G$ consists of $x^m$, $m \in \mathbb{Z}$.*

**Example 2.5.** *Let $G = \mathbb{Z}$ be a group with addtion. What is $\langle 2 \rangle$?*

**Proposition 2.6.** *Let $G$ a group and $x \in G$. The cyclic group $\langle x \rangle$ is the smallest subgroup containing $x$.*

**Definition 2.7.** *Let $G$, $H$ be groups and $\varphi : G \to H$ a map. The map $\varphi$ is called a homomorphism if it preserves the group structure, in a sense that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$.*

**Proposition 2.8.** *If $\varphi$ is a homomorphism from $G$ to $H$, then it maps the identity and inverses in $G$ to the inverse and inverses in $H$.*

**Example 2.9.** *Consider a map $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $x \mapsto 2x$. One can see that $\varphi$ is a homomorphism.*

**Definition 2.10.** *An isomorphism is a bijective homomorphism. We say two groups are isomorphic if there is an isomorphism between the groups.*

## 3. Groups of transformations

Let $\mathcal{V}$ a set. We have seen that the set of all bijections $\alpha : \mathcal{V} \to \mathcal{V}$ with compositions is a group, say $G$. In this section, we will see several subgroups of $G$. In particular, we are interested in the case $\mathcal{V} = \mathbb{R}^2$.

**Theorem 3.1.** *Let $\mathcal{T}$ be the set of all translations $\tau_A : \mathbb{R}^2 \to \mathbb{R}^2$, $A \in \mathbb{R}^2$. Then, $\mathcal{T}$ with compositions is a group.*

*Proof.* By the lemma, it suffices to show that for any $A, B \in \mathbb{R}^2$, $\tau_A \tau_B^{-1} \in \mathcal{T}$. ■

**Definition 3.2.** *We call a bijection map $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ a dilatation if $\alpha$ is either a translation or a central dilatation.*

**Definition 3.3.** *We call a bijection map collineation if it maps a line to a parallel line.*

As we have seen before, dilatations are collineation.

**Theorem 3.4.** *The set of all dilatations $\mathcal{D}$ with compositions forms a group.*

*Proof.* Since $(\tau_A)^{-1} = \tau_{-A}$ and $(\delta_{C,r})^{-1} = \delta_{C,1/r}$, it suffices to show that

(1) $\tau_A \tau_B \in \mathcal{D}$,
(2) $\tau_A \delta_{C,r} \in \mathcal{D}$,
(3) $\delta_{C,r} \tau_A \in \mathcal{D}$,
(4) $\delta_{C,r} \delta_{D,s} \in \mathcal{D}$.

We know that (1) is trivial and (3) follows from (2). Thus, it is enough to show (2) and (4). We first have

$$\begin{aligned} \tau_A \delta_{C,r}(X) &= \tau_A((1-r)C + rX) \\ &= (1-r)(\frac{1}{1-r}A + C) + rX \\ &= \delta_{P,r}(X) \end{aligned}$$

where $P = \frac{1}{1-r}A + C$. Also,

$$\delta_{C,r}\delta_{D,s}(X) = (1-r)C + r(1-s)D + rsX = \begin{cases} \tau_{(1-r)(C-D)}(X), & rs = 1, \\ \delta_{Q,rs}(X), & rs \neq 1, \end{cases}$$

where $Q = (1-rs)^{-1}((1-r)C + r(1-s)D)$. ■

## References

[T] Philippe Tondeur, *Vectors and Transformations in Plane Geometry*, Publish Or Perish, Inc. 1993

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