MATH 403 FALL 2021: HOMEWORK 10 SOLUTION

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Note: The problems 2, 3, 5, 8, 10 will only be graded.

1. Let α be an isometry and ℓ be a line. Show that $\sigma_{\alpha(\ell)} = \alpha\sigma_\ell\alpha^{-1}$.

Solution: Let $\beta = \alpha\sigma_\ell\alpha^{-1}$. Note that β is not the identity. Since $\beta^2 = (\alpha\sigma_\ell\alpha^{-1}) \circ (\alpha\sigma_\ell\alpha^{-1}) = \alpha\sigma_\ell\alpha^{-1} = \beta$, it is involutive. Let $Y \in \alpha(\ell)$, then there exists $X \in \ell$ such that $\alpha(X) \in \alpha(\ell)$. Since $\sigma_\ell(X) = X$, we have

$$\beta(Y) = \alpha\sigma_\ell\alpha^{-1}(Y) = \alpha\sigma_\ell\alpha^{-1}(\alpha(X)) = \alpha\sigma_\ell(X) = \alpha(X) = Y.$$

Therefore, β is the reflection in $\sigma(\ell)$.

2. Exercise 4.2

Solution: Let $\gamma = \alpha^{-1} \circ \beta$, then

$$\gamma(A) = A, \quad \gamma(B) = B, \quad \gamma(C) = C.$$

Since A, B, C are collinear, we have $\gamma = \mathrm{Id}$, which implies $\alpha = \beta$.

3. Exercise 4.5: Let α be an isometry. Suppose that $\alpha\sigma_\ell = \sigma_\ell\alpha$ holds for any line ℓ. Show that α is the identity map.

Solution: Let $X \in \mathbb{R}^2$. Choose two different lines ℓ_1, ℓ_2 such that both lines contain X. For each $i = 1, 2$,

$$\alpha(X) = \alpha\sigma_{\ell_i}(X) = \sigma_{\ell_i}\alpha(X) = \sigma_{\ell_i}(\alpha(X)),$$

which implies that $\alpha(X)$ is a fixed point of σ_{ℓ_i}. Thus, $\alpha(X) \in \ell_1 \cap \ell_2$. Therefore, we conclude that $\alpha(X) = X$, as desired.

4. Exercise 4.7: Let ℓ_1, ℓ_2 be lines with $\ell_1 \perp \ell_2$. Show that $\sigma_{\ell_1}\sigma_{\ell_2} = \sigma_C$ for some $C \in \mathbb{R}^2$.

Solution: Let $X \in \mathbb{R}^2$, $Y = \sigma_{\ell_2}(X)$, and $Z = \sigma_{\ell_1}(Y)$. Since $|C - X| = |C - Y| = |C - Z|$, we see that C is the circumcenter of $\triangle XYZ$. Since ℓ_1 is perpendicular to ℓ_2, $\triangle XYZ$ is a right triangle. Thus, C is the midpoint of the hypotenuse, that is, $C = \frac{1}{2}(X + Z)$.

5. Exercise 4.9

Solution: We have

$$\begin{aligned} & \sigma_\ell\sigma_B\sigma_\ell\sigma_A = \mathrm{Id} \\ \Leftrightarrow \quad & \sigma_{\sigma_\ell(B)} = \sigma_A \\ \Leftrightarrow \quad & \sigma_\ell(B) = A. \end{aligned}$$

By the definition of σ_ℓ, the last statement is equivalent to that ℓ is the perpendicular bisector of $\overline{AB}$.

6. Exercise 4.10

Solution: We have

$$\begin{aligned} & \sigma_B \sigma_\ell \sigma_B \sigma_A \sigma_\ell \sigma_A = \mathrm{Id} \\ \Leftrightarrow \quad & \sigma_{\sigma_B(\ell)} = \sigma_{\sigma_A(\ell)} \\ \Leftrightarrow \quad & \sigma_B(\ell) = \sigma_A(\ell) \\ \Leftrightarrow \quad & \ell = \sigma_A \sigma_B(\ell). \end{aligned}$$

Suppose ℓ is parallel to $\overline{AB}$ and $X \in \ell$. Let $Y = \sigma_B \sigma_A(X)$. Since

$$X - Y = X - (2B - (2A - X)) = 2(A - B),$$

$Y \in \ell$. Since $\sigma_A \sigma_B(Y) = X$, we conclude that $\ell \subseteq \sigma_A \sigma_B(\ell)$. Similarly, one can show the opposite direction and conclude that $\ell = \sigma_A \sigma_B(\ell)$.

Suppose $\ell = \sigma_A \sigma_B(\ell)$. Let $X \in \ell$. If $X = \sigma_A \sigma_B(X)$, then $A = B$, which is a contradiction. Thus, $Y := \sigma_A \sigma_B(X) \in \ell$ and $X \neq Y$. Since

$$X - Y = X - (2A - (2B - X)) = -2(A - B),$$

we conclude that ℓ is parallel to $\overline{AB}$.

7. Exercise 4.11

Solution: Let ℓ be the line perpendicular to m and n and $M \in \ell \cap m$ and $N \in \ell \cap n$. Then, we know that

$$\sigma_n \sigma_m = \tau_{2(N-M)}, \quad \sigma_m \sigma_n = \tau_{2(M-N)}.$$

Thus, $\sigma_n \sigma_m = \sigma_m \sigma_n$ if and only if $\tau_{2(N-M)} = \tau_{2(M-N)}$ iff $N = M$ iff $m = n$.

8. Exercise 4.12

Solution: For any $X \in \mathbb{R}^2$,

$$X + \tau_A \sigma_P(X) = X + A + (2P - X) = A + 2P.$$

Thus,

$$\tau_A \sigma_P(X) = (A + 2P) - X = \sigma_C(X)$$

where $C = \frac{1}{2}(A + 2P)$. The second assertion is similar.

9. Exercise 4.13

Solution: By Exerise 4.12, we know that $\tau_A = \sigma_P \sigma_Q$ where $P \in \mathbb{R}^2$ and $Q = \frac{1}{2}(2P - A)$. Then,

$$\alpha \tau_A \alpha^{-1} = \alpha \sigma_P \sigma_Q \alpha^{-1} = \alpha \sigma_P \alpha^{-1} \alpha \sigma_Q \alpha^{-1} = \sigma_{\alpha(P)} \sigma_{\alpha(Q)} = \tau_R$$

where $R = 2(\alpha(P) - \alpha(Q))$. If α is linear isometry, then

$$R = 2(\alpha(P) - \alpha(Q)) = \alpha(2P - 2Q) = \alpha(A).$$

10. Let α be a rotation with center $C \in \mathbb{R}^2$ through the oriented angle $\theta \in (-\pi, \pi]$ and ℓ be a line. Suppose α is not the identity map and $\alpha(\ell) = \ell$. Show that α is the central reflection in C.

Solution: This is Theorem 4.32 (pg. 96).