MATH 403 FALL 2021: HOMEWORK 7 SOLUTION

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1. Exercise 3.3

Solution: Without loss of generality, we consider $\triangle OAB$ with $|A - A'| = |B - B'|$, $A' = B/2$, $B' = A/2$. Thus,

$$|A - A'|^2 = |A|^2 - A \cdot B + |B|^2/4 = |B|^2 - A \cdot B + |A|^2/4 = |B - B'|^2,$$

which implies $|A|^2 = |B|^2$ as desired.

2. Exercise 3.4

Solution: Suppose $A \cdot X$ for all $X \in \mathbb{R}^2$. In particular, if $X = A$, then $|A|^2 = 0$. We have seen that $|A| = 0$ if and only if $A = O$, which completes the proof.

3. Exercise 3.5

Solution: Suppose $A, B \neq O$ and $A \cdot B = 0$. Assume that $O \in \ell_{AB}$, then there exists $a, b \in \mathbb{R}$ such that $a + b = 1$ and $aA + bB = O$. Since either $a \neq 0$ or $b \neq 0$, suppose $a \neq 0$ WLOG. Then, $A = -\frac{b}{a}B$. Thus,

$$A \cdot B = -\frac{b}{a}|B|^2.$$

Since $B \neq O$, we have $b = 0$. But, $aA + bB = aA = O$ implies that $A = O$ which contradicts to the assumption. Thus, $O \notin \ell_{AB}$.

4. Exercise 3.7

Solution: Let H be the orthocenter and K be the circumcenter. We have seen that $H = \delta_{G,-2}(K)$, where G is the centroid. Thus,

$$H = (1 - (-2))G - 2K = A + B + C - 2K.$$

Since

$$A' + A'' = \frac{1}{2}(B + C) + \frac{1}{2}(H + A) = \frac{1}{2}(A + C) + \frac{1}{2}(H + B) = B' + B'',$$

we see that $A''B''A'B'$ is a parallelogram. Also,

$$|B' - B''| = |\frac{1}{2}(A + C) - \frac{1}{2}(H + B)| = |B - K|,$$
$$|A' - A''| = |\frac{1}{2}(B + C) - \frac{1}{2}(H + A)| = |A - K|.$$

Since K is the circumcenter, $|B - K| = |A - K|$. Since the diagonals have the same length, $A''B''A'B'$ is a rectangle.

5. Exercise 3.9

Solution: If X satisfies $|X-A| = r|X-B|$, then

$$|X|^2 - 2A\cdot X + |A|^2 = r^2(|X|^2 - 2B\cdot X + |B|^2),$$

$$(1-r^2)|X|^2 - 2(A-r^2B)\cdot X + (1-r^2)^{-1}|A-r^2B|^2 = r^2|B|^2 - |A|^2 + (1-r^2)^{-1}|A-r^2B|^2,$$

$$(1-r^2)|X-(1-r^2)^{-1}(A-r^2B)|^2 = \frac{r^2}{1-r^2}|A-B|^2.$$

Therefore, X belongs to the circle with radius $r|1-r^2|^{-1}|A-B|$ and center $(1-r^2)^{-1}(A-r^2B)$.